Bilateral trading and incomplete information: The Coase conjecture in a small market.

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Abstract

We study a model of decentralised bilateral interactions in a small market where one of the sellers has private information about her value. There are two identical buyers and another seller, apart from the informed one, whose valuation is commonly known to be in between the two possible valuations of the informed seller. This represents an attempt to model alternatives to current partners on both sides of the market. We consider an infinite horizon game with simultaneous one-sided offers and simultaneous responses. We show that as the discount factor goes to 1, the outcome of any stationary PBE of the game is *unique* and prices in all transactions converge to the same value. We then *characterise* one such PBE of the game.

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1 Introduction

This paper studies a small market in which one of the players has private information about her valuation. As such, it is a first step in combining the literature on (bilateral) trading with incomplete information with that on market outcomes obtained through decentralised bilateral bargaining.

We shall discuss the relevant literature in detail later on in the introduction. Here we summarise the motivation for studying this problem.

One of the most important features in the study of bargaining is the role of "outside options" in determining the bargaining solution. There have been several different approaches modelling what these options are, starting with treating alternatives to the current bargaining game as exogenously given and always available. Accounts of negotiation directed towards practitioners and policy-oriented academics, like Raiffa's masterly "The Art and Science of Negotiation",([34]) have emphasised the key role of the "Best Alternative to the Negotiated Agreement" and mentioned the role of searching for such alternatives in preparing for negotiations. Search for outside options has also been considered, as well as search for bargaining partners in a general coalition formation context.

Proceeding more or less in parallel, there has been considerable work on bargaining with incomplete information. The major success of this work has been the complete analysis of the bargaining game in which the seller has private information about the minimum offer she is willing to accept and the buyer, with only the common knowledge of the probability distribution from which the seller's reservation price is drawn, makes repeated offers which the seller can accept or reject; each rejection takes the game to another period and time is discounted at a common rate by both parties[1]. With the roles of the seller and buyer reversed, this has also been part of the development of the foundations of dynamic monopoly and the Coase conjecture [2].

One question that naturally arises is: does the Coase conjecture in bilateral bargaining with incomplete information and one-sided offers continue to hold in the presence of outside options? In a recent paper, Board and Pycia([6]) have given a negative answer to this question. They consider two settings. In both of them, a *responder* has the option of calling

[1] Other, more complicated, models of bargaining have also been formulated (for example, [10]), with two-sided offers and two-sided incomplete information, but these have not usually yielded the clean results of the game with one-sided offers and one-sided incomplete information.

[2] The "Coase conjecture" relevant here is the bargaining version of the dynamic monopoly problem, namely that if an uninformed seller (who is the only player making offers) has a valuation strictly below the informed buyer's lowest possible valuation, the unique sequential equilibrium as the seller is allowed to make offers frequently, has a price that converges to the lowest buyer valuation. Here we show that even if one adds endogenous outside options for both players, a similar conclusion holds for all stationary equilibria-hence an extended Coase conjecture holds.

off the negotiation with its current partner and can opt for an outside option. In their first setting, the responder takes a fixed outside option and they obtain a unique equilibrium in which the seller charges the constant monopoly price. The presence of outside options makes a buyer with low valuation leave the market. Hence, a sustained high-price equilibrium can be supported since low value buyers expecting the price to be high take the outside option; thus there is positive selection in the demand pool. In the second specification, the outside option is obtained through a draw from a given distribution. The buyer's outside option is constituted by the expected value of his future search opportunities. Hence, the monopoly price result of the first setting can be applied.

We revisit the question posed by Board and Pycia with a somewhat different formulation of "outside options". The usual model of an outside option treats it as a payoff obtained from some external opportunity, either given at the beginning of the game or obtainable through search. We seek to model these alternatives explicitly and as the result of a strategic choice made by the players. Thus, though trades remain bilateral, a buyer can choose to make an offer to a different seller than the one who rejected his last offer and the seller can entertain an offer from some buyer whom she has not bargained with before. These alternatives are internal to the model of a small market, rather than given as part of the environment. What we do is as follows: We take the basic problem of a seller with private information and an uninformed buyer and add another buyer-seller pair; here the new seller's valuation is commonly known and is different from the possible valuations of the informed seller[3]. The buyers' valuations are identical and commonly known. Specifically, the informed seller's valuation can either be L or H ($H > L \geq 0$) and the new seller's valuation is M such that $M \in (L, H)$. Each seller has one good and each buyer wants at most one good. This is the simplest extension of the basic model that gives rise to outside options for each player, though unlike the literature on exogenous outside options, only *one* buyer can deviate from the incomplete information bargaining to take his outside option with the other seller (if this other seller accepts the offer), since each seller only has one good to sell.[4]

In our model, buyers make offers *simultaneously*, each buyer choosing only one seller.[5] Sellers also respond simultaneously, accepting at most one offer. A buyer whose offer is

[3]When we consider a continuum of possible valuations for the seller, the valuation of the known seller is one of them.

[4]What do the seller's valuations represent? (The buyers' valuations are clear enough.) We could consider a seller can produce a good, if contracted to do so, at a private cost of H or L and pays no cost otherwise. Or one could consider the value she gets from keeping the object as H or L. Thus, supposing her value is L, if she accepts a price offer p with probability α, her payoff is $L(1-\alpha) + (p)\alpha = (p - L)\alpha + L$.. Hence, one can think of $(p - L)$ as the net benefit to the seller from selling the good at price p. For the purpose of making the decision on whether to accept or reject, the two interpretations give identical results.

[5]Simultaneous offers extensive forms probably capture best the essence of competition .

accepted by a seller leaves the market with the seller and the remaining players play the one-sided offers game with or without asymmetric information. We consider the case when buyers' offers are public, so that the continuation strategies can condition on both offers in a given period and the set of players remaining[6]. The **main result** of our analysis shows that in the incomplete information game, **any stationary equilibrium** must have certain specific qualitative features. As agents become patient enough, these qualitative features enable us to show that **all price offers in any stationary equilibrium converge to the highest possible value of the informed seller** (H). We then **characterise** one such stationary equilbrium. Unlike the two-player case, where there is a unique sequential equilibrium for the "gap" case, there could be non-stationary equilibria with different outcomes in the four-player, public offers case, though there is a unique public perfect Bayesian equilibrium outcome with private offers.[7] We note that the discussion of the features of the stationary equilibrium, if one exists, is concerned with the properties of proposed actions occurring with positive probability on the equilibrium path. To complete the analysis, we need to show existence and here the properties of outcome paths following deviations becomes important.

The equilibrium we construct to demonstrate existence is in (non-degenerate) randomized behavioral strategies (as in the two-player game). As agents become patient enough, in equilibrium competition always takes place for the seller whose valuation is commonly known. The *equilibrium* behavior of beliefs is similar to the two-player asymmetric information game and the same across public and private offers. However, the off-path behaviour sustaining any equilibrium is different and has to take into account many more possible deviations.

The result of this paper is not confined to uncertainty described by two types of seller. Even if the informed seller's valuation is drawn from a continuous distribution on $(L, H]$, we show that the asymptotic convergence to H still holds as the unique limiting stationary equilibrium outcome.

Related literature: The modern interest in this approach dates back to the seminal

[6]We also discuss private offers, in the extensions,, i.e when only the proposer and the recipient of an offer know what it is and the only public information is the set of players remaining in the game.

[7]In the complete information case (see Chatterjee and Das 2015), we get a similar result. But this does not mean the analyses are the same. In the bilateral bargaining game with complete information where the seller has valuation H, the price is H; if it is L, the price is L. From this fact, it is non-trivial to guess that the Coase conjecture is true, namely that for the discount factor going to 1 and the probability of a H seller being positive the price goes to H. (This explains the large number of papers on this bilateral case.) With four players, even with only one seller's value being unknown, the problem is compounded by the presence of the other alternatives. We leave out the construction of the equilibrium itself, which requires some careful consideration of appropriate beliefs. Without this construction, of course, the equilibrium path cannot be known to be such, so the fact that two equilibrium paths end up looking similar doesn't mean that the *equilibria* are the same.

work of Rubinstein and Wolinsky ([35], [36]), Binmore and Herrero ([5])and Gale ([17]),[18]). These papers, under complete information, mostly deal with random matching in large anonymous markets, though Rubinstein and Wolinsky (1990) is an exception. Chatterjee and Dutta ([8]) consider strategic matching in an infinite horizon model with two buyers and two sellers and Rubinstein bargaining, with complete information. In a companion paper ([7]), we analyse markets under complete information where the bargaining is with one-sided offers.

There are several papers on searching for outside options, for example, Chikte and Deshmukh ([12]), Muthoo ([29]), Lee ([28]), Chatterjee and Lee ([11]). Chatterjee and Dutta ([9]) study a similar setting as this paper but with sequential offers by buyers.

A rare paper analysing outside options in asymmetric information bargaining is that by Gantner([22]), who considers such outside options in the Chatterjee and Samuelson ([10]) model. Our model differs from hers in the choice of the basic bargaining model and in the explicit analysis of a small market with both public and private targeted offers. (There is competition for "outside options" too, in our model but not in hers.) Another paper, which in a completely different setting, discusses outside options and bargaining is Atakan and Ekmekci([1]). Their model is based on the presence of inflexible behavioral types and matching over time. They consider the steady state equilibria of this model in which there are inflows of different types of agents every period. Their main result shows that there always exist equilibria where there are selective breakups and delay, which in turn leads to inefficiency in bargaining.

Some of the main papers in one-sided asymmetric information bargaining are the well-known ones of Sobel and Takahashi([38]), Fudenberg, Levine and Tirole ([15]), Ausubel and Deneckere ([2]). The dynamic monopoly papers mentioned before are the ones by Gul and Sonnenschein ([23]) and Gul, Sonnenschein and Wilson([24]).[8]

There are papers in very different contexts that have some of the features of this model. For example, Swinkels [40] considers a discriminatory auction with multiple goods, private values (and one seller) and shows convergence to a competitive equilibrium price for fixed supply as the number of bidders and objects becomes large. We keep the numbers small, at two on each side of the market. Other papers which have looked into somewhat related issues but in a different environment are Fuchs and Skrzypacz ([14]), Kaya and Liu ([27]) and Horner and Vieille ([26]). We do not discuss these in detail because they are not directly comparable to our work.

Outline of rest of the paper. The rest of the paper is organised as follows. Section 2 discusses the model in detail. The qualitative nature of the equilibrium and its detailed

[8]See also the review paper of Ausubel, Cramton and Deneckere ([3]).

derivation is given in section 3, which is the heart of the paper. The asymptotic characteristics of the equilibrium are obtained in Section 4. Section 5 discusses the possibility of other equilibria, as well as the private offers case. Finally, Section 6 concludes the paper.

In Appendix (H) we discuss in detail a model where the informed seller's valuation is drawn from a continuous distribution on $(L, H]$.

2 The Model

2.1 Players and payoffs

The setup we consider has two uninformed homogeneous buyers and two heterogeneous sellers. Buyers (B_1 and B_2) have a common valuation of v for the good (the maximum willingness to pay for a unit of the indivisible good). There are two sellers. Each of the sellers owns one unit of the indivisible good. Sellers differ in their valuations. The first seller (S_M) has a reservation value of M which is commonly known. The other seller (S_I) has a reservation value that is private information to her. S_I's valuation is either L or H, where,

$$v > H > M > L$$

It is commonly known by all players that the probability that S_I has a reservation value of L is $\pi \in [0, 1)$. It is worthwhile to mention that $M \in [L, H]$ constitutes the only interesting case. If $M < L$ (or $M > H$) then one has no uncertainty about which seller has the lowest reservation value. Although our model analyses the case of $M \in (L, H)$, the same asymptotic result will be true for $M \in [L, H]$.

Players have a common discount factor $\delta \in (0, 1)$. If a buyer agrees on a price p^j with seller S_j at a time point t, then the buyer has an expected discounted payoff of $\delta^{t-1}(v - p^j)$. The seller's discounted payoff is $\delta^{t-1}(p^j - u_j)$, where u_j is the valuation of seller S_j.

2.2 The extensive form

This is an infinite horizon, multi-player bargaining game with one-sided offers and discounting. The extensive form is as follows:

At each time point $t = 1, 2, ..$, offers are made simultaneously by the buyers. The offers are targeted. This means an offer by a buyer consists of a seller's name (that is S_I or S_M) and a price at which the buyer is willing to buy the object from the seller he has chosen. Each buyer can make only one offer per period. Two informational structures can be considered; one in which each seller observes all offers made (*public targeted offers*) and the one (

private targeted offers) in which each seller observes only the offers she gets. (Similarly for the buyers, after the offers have been made-in the private offers case each buyer knows his own offer and can observe who leaves the market.) In the present section we shall focus on the first and consider the latter in a subsequent section. A seller can accept at most one of the offers she receives. Acceptances or rejections are simultaneous. Once an offer is accepted, the trade is concluded and the trading pair leaves the game. Leaving the game is publicly observable. The remaining players proceed to the next period in which buyers again make price offers to the sellers. As is standard in these games, time elapses between rejections and new offers.

3 Equilibrium

We will look for *Perfect Bayes Equilibrium*[16] of the above described extensive form. This requires sequential rationality at every stage of the game given beliefs and the beliefs being compatible with Bayes' rule whenever possible, on and off the equilibrium path. We will mostly focus on *stationary* equilibria. These are the equilibria, where strategies depend on the history only to the extent to which it is reflected in the updated value of π (the probability that S_I's valuation is L). Thus, at each time point, buyers' offers depend only on the number of players remaining and the value of π. The sellers' responses depend on the number of players remaining, the value of π and the offers made by the buyers.

3.1 The Benchmark Case: Complete information

Before we proceed to the analysis of the incomplete information framework, we state the results of the above extensive form with complete information. A formal analysis of the complete information framework has been done in a companion paper [7].

Suppose the valuation of S_I is commonly known to be H. In that case there exists a unique[9] stationary equilibrium (an equilibrium in which buyers' offers depend only on the set of players present and the sellers' responses depend on the set of players present and the offers made by the buyers) in which one of the buyers (say B_1) makes offers to both the sellers with positive probability and the other buyer (B_2) makes offers to S_M only. Suppose $E(p)$ represents the expected maximum price offer to S_M in equilibrium. Assuming that there exists a unique $p_l \in (M, H)$ such that,

$$p_l - M = \delta(E(p) - M)^{10}$$

[9]Up to the choice of B_1 and B_2

[10]Given the nature of the equilibrium it is evident that $M(p_l)$ is the minimum acceptable price for S_M

, the equilibrium is as follows:

1. B_1 offers H to S_I with probability q. With the complementary probability he makes offers to S_M. While offering to S_M, B_1 randomises his offers using an absolutely continuous distribution function $F_1(.)$ with $[p_l, H]$ as the support. F_1 is such that $F_1(H) = 1$ and $F_1(p_l) > 0$. This implies that B_1 puts a mass point at p_l.

2. B_2 offers M to S_M with probability $q^{'}$. With the complementary probability his offers to S_M are randomised using an absolutely continuous distribution function $F_2(.)$ with $[p_l, H]$ as the support. $F_2(.)$ is such that $F_2(p_l) = 0$ and $F_2(H) = 1$.

It is shown in [7] that this p_l exists and is unique. Also, the outcome implied by the above equilibrium play constitutes the unique stationary equilibrium outcome and as $\delta \to 1$,

$$q \to 0 \ , \ q^{'} \to 0 \text{ and } p_l \to H$$

This means that as market frictions go away, we tend to get a uniform price in different buyer-seller matches. In this paper, we show a similar asymptotic result even with incomplete information, with a different analysis.

3.2 Equilibrium of the one-sided incomplete information game with two players

The equilibrium of the whole game contains the analyses of the different two-player games as essential ingredients. If a buyer-seller pair leaves the market after an agreement and the other pair remains, we have a continuation game that is of this kind. We therefore first review the features of the two-player game with one-sided private information and one-sided offers.

The setting is as follows: There is a buyer with valuation v, which is common knowledge. The seller's valuation can either be H or L where $v > H > L = 0$[11]. At each period, conditional on no agreement being reached till then, the buyer makes the offer and the seller (informed) responds to it by accepting or rejecting. If the offer is rejected then the value of π is updated using Bayes' rule and the game moves on to the next period when the buyer again makes an offer. This process continues until an agreement is reached. The equilibrium of this game(as described in, for example, [13]) is as follows.

For a given δ, we can construct an increasing sequence of probabilities, $d(\delta) = \{0, d_1, ..., d_t, ...\}$ so that for any $\tilde{\pi} \in (0, 1)$ there exists a $t \geq 0$ such that $\tilde{\pi} \in [d_t, d_{t+1})$. Suppose at a particular time point, the play of the game so far and Bayes' Rule implies that the updated belief is

when she gets one(two) offer(s).

[11] $L = 0$ is assumed to simplify notations and calculations.

π. Thus, there exists a $t \geq 0$ such that $\pi \in [d_t, d_{t+1})$. The buyer then offers $p_t = \delta^t H$. The H type seller rejects this offer with probability 1. The L type seller rejects this offer with a probability that implies, through Bayes' Rule, that the updated value of the belief $\pi^u = d_{t-1}$. The cutoff points d_t's are such that the buyer is indifferent between offering $\delta^t H$ and continuing the game for a maximum of t periods from now or offering $\delta^{t-1}H$ and continuing the game for a maximum of $t-1$ periods from now. Thus, here t means that the game will last for at most t periods from now. The maximum number of periods for which the game can last is given by $N(\delta)$. It is already shown in [13] that this $N(\delta)$ is uniformly bounded above by a finite number N^* as $\delta \to 1$.

Since we are describing a PBE for the game it is important that we specify the off-path behavior of the players. First, the off-path behavior should be such that it sustains the equilibrium play in the sense of making deviations by the other player unprofitable and second, if the other player has deviated, the behavior should be equilibrium play in the continuation game, given beliefs. We relegate these discussion to appendix (A).

Given a π, the expected payoff to the buyer $v_B(\pi)$ is calculated as follows:

For $\pi \in [0, d_1)$, the two-player game with one-sided asymmetric information involves the same offer and response as the complete information game between a buyer of valuation v and a seller of valuation H. Thus we have

$$v_B(\pi) = v - H \text{ for } \pi \in [0, d_1)$$

For $\pi \in [d_t, d_{t+1})$, $(t \geq 1\ )$, we have,

$$v_B(\pi) = (v - \delta^t H)a(\pi, \delta) + (1 - a(\pi, \delta))\delta(v_B(d_{t-1})) \tag{1}$$

where $a(\pi, \delta)$ is the equilibrium acceptance probability of the offer $\delta^t H$.

These values will be crucial for the construction of the equilibrium of the four-player game. However, before we show the existence of a stationary equilibrium of the four player game, we show that if a stationary equilibrium exists, then the qualitative nature of the equilibrium is unique and also as $\delta \to 1$, outcome of any stationary equilibrium is unique. This is described in the following subsection.

3.3 Uniqueness of the asymptotic equilibrium outcome

In this subsection, we show that prices in all stationary equilibrium outcomes, if a stationary equilibrium exists, must converge to the same value as $\delta \to 1$. This, together with the construction of a stationary equilibrium elsewhere in the paper (showing existence constructively), shows that there is a unique limiting stationary equilibrium outcome. The main result of this section is summarised in theorem (1). First, we prove the following proposition which establishes the main result conditional on a particular kind of equilibrium being ruled out. Later, we prove that this particular kind of equilibrium never exists.

Proposition 1 *Consider the set of stationary equilibria of the four player game such that any equilibrium belonging to this set has the property that both buyers do not make offers only to the informed seller (S_I) on the equilibrium path. As the discount factor $\delta \to 1$, all price offers in any equilibrium belonging to this set converge to H*

Proof. We prove this proposition in steps, through a series of lemmas. First, we show that for any equilibrium belonging to the set of equilibria considered, the following lemma holds.

Lemma 1 *For any $\pi \in (0,1)$, it is never possible to have a stationary equilibrium in the set of equilibria considered such that both buyers offer only to S_M on the equilibrium path.*

Proof. Suppose it is the case that there exists a stationary equilibrium in the game with four players such that both buyers offer only to S_M. Both buyers should have a distribution of offers to S_M with a common support[12] $[\underline{s}(\pi), \bar{s}(\pi)]$. The payoff to each buyer should then be $(v - \bar{s}(\pi)) = v_4(\pi)$(say). Let $v_B(\pi)$ be the payoff obtained by a buyer when his offer to S_M gets rejected. This is the payoff a buyer obtains by making offers to the informed seller in a two player game.

Consider any $s \in [\underline{s}(\pi), \bar{s}(\pi)]$ and one of the buyers (say B_1). If the distributions of the offers are given by F_i for buyer i, then we have

$$(v - s)F_2(s) + (1 - F_2(s))\delta v_B(\pi) = v - \bar{s}(\pi)$$

This follows from the buyer B_1's indifference condition[13].

[12] If the upper bounds are not equal, then the buyer with the higher upper bound can profitably deviate. On the other hand, if the lower bounds are different, then the buyer with the smaller lower bound can profitably deviate.

[13] If there exists a stationary equilibrium where both buyers offer to S_M only, then the lower bound of the common support of offers is not less than the minimum acceptable price to S_M in the candidate stationary equilibrium. To see this, let $p_2^2(\pi) = (1-\delta)M + \delta E_p^2(\pi)$. Suppose the lower bound of the support is strictly less than $p_2^2(\pi)$. Let $z(\pi)$ be the probability with which each buyer's offer is strictly less than $p_2^2(\pi)$. If $v_4^2(\pi)$ is the payoff to the buyers in this candidate equilibrium, the expected payoff to the buyer from making an offer strictly less than $p_2^2(\pi)$ is $z(\pi)\delta v_4^2(\pi) + (1 - z(\pi))\delta v_B(\pi)$. In equilibrium, we must have $z(\pi)\delta v_4^2(\pi) + (1 - z(\pi))\delta v_B(\pi) = v_4^2(\pi)$. Either $v_B(\pi) > v_4^2(\pi)$ or $v_B(\pi) \le v_4^2(\pi)$. In the former case the equality does not hold for values of δ close to 1 and in the later case the equality does not hold for any value of $\delta < 1$.

Since in equilibrium, the above needs to be true for any $s \in [\underline{s}(\pi), \bar{s}(\pi)]$, we must have $v - \bar{s}(\pi) > \delta v_B(\pi)$. The above equality then gives us

$$F_2(s) = \frac{(v - \bar{s}(\pi)) - \delta v_B(\pi)}{(v - s) - \delta v_B(\pi)}$$

Since $v - \bar{s}(\pi) > \delta v_B(\pi)$, for $s \in [\underline{s}(\pi), \bar{s}(\pi))$, we have $v - s > v - \bar{s}(\pi) > \delta v_B(\pi)$. This would imply

$$F_2(\underline{s}(\pi)) > 0$$

Similarly, we can show that

$$F_1(\underline{s}(\pi)) > 0$$

In equilibrium, it is not possible for both the buyers to put mass points at the lower bound of the support. Hence, S_M cannot get two offers with probability 1. This concludes the proof of the lemma. ■

For any equilibrium belonging to the set of equilibria we are considering, we know that S_M must get at least one offer with positive probability. The above lemma implies that S_I also gets at least one offer with a positive probability. We will now argue that for any equilibrium in the set of equilibria considered, S_M always accepts an equilibrium offer immediately. This is irrespective of whether S_M gets one offer or two offers.

To show this formally, consider such an equilibrium. We first define the following. Given a π, let $p_i(\pi)$ be the minimum acceptable price to the seller S_M in the event she gets i ($i = 1, 2$) offer(s) in the considered equilibrium. We have

$$p_1(\pi) - M = (1 - (\alpha(\pi))\delta[E_p(\tilde{\pi}) - M]$$

$E_p(\tilde{\pi})$ is the price corresponding to the expected equilibrium payoff to the seller S_M in the event she rejects the offer and the informed seller does not accept the offer. It is evident that when the seller S_M is getting one offer, the informed seller is also getting an offer. Here $\alpha(\pi)$ is the probability with which the informed seller accepts the offer and $\tilde{\pi}$ is the updated belief.

Similarly, we have

$$p_2(\pi) - M = \delta[E_p(\pi) - M]$$

where $E_p(\pi)$ is the price corresponding to the expected equilibrium payoff to S_M in the event she rejects both offers. In appendix I we argue that $E_p(\pi) > M$. The following lemma has the consequence that S_M always accepts an equilibrium offer (or highest of the equilibrium offers) immediately.

Lemma 2 *For any $\pi < 1$, if we restrict ourselves to the set of equilibria considered, then in any arbitrary equilibrium, it is never possible for a buyer to make an offer to S_M, which is strictly less than* $\min\{p_1(\pi), p_2(\pi)\}$.

Proof. Suppose the conclusion of the lemma does not hold, so there is such an equilibrium. Let the payoff to the buyers from this candidate equilibrium of the four-player game be $v_4(\pi)$. In appendix J we argue that $v_4(\pi) < v - p_2(\pi)$. Let $v_B(\pi)$ be the payoff the buyer gets by making offers to S_I in a two-player game.

Consider the buyer who makes the lowest offer to S_M. We label this buyer as B_1 and the lowest offer as $\underline{p(\pi)}$,where $\underline{p(\pi)} < \min\{p_1(\pi), p_2(\pi)\}$. Let $q(\pi)$ be the probability with which the other buyer makes an offer to the seller S_I. Let $\gamma(\pi)$ be the probability with which the other buyer, conditional on making offers to the seller S_M, makes an offer which is less than $p_2(\pi)$. Finally, $\alpha(\pi)$ is the probability with which the informed seller accepts an offer if the other buyer makes an offer to her. Since B_1's offer of $\underline{p(\pi)}$ to S_M is always rejected, the payoff to B_1 from making such an offer is

$$\{q(\pi)\delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E^b_p(\tilde{\pi}))\} + (1 - q(\pi))\delta\{\gamma(\pi)v_4(\pi) + (1 - \gamma(\pi))v_B(\pi)\}$$

where $E^b_p(\tilde{\pi})$ is such that $(v - E^b_p(\tilde{\pi}))$ is the expected equilibrium payoff to the buyer if the updated belief is $\tilde{\pi}$. We first argue that $(v - E^b_p(\tilde{\pi}))$ is less than or equal to $(v - E_p(\tilde{\pi}))$. This is because since $(E_p(\tilde{\pi}) - M)$ is the expected equilibrium payoff to the seller S_M when the belief is $\tilde{\pi}$, there is at least one price offer by the buyer, which is greater than or equal to $E_p(\tilde{\pi})$. Hence, we have

$$\delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E^b_p(\tilde{\pi}))\} \leq \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p(\tilde{\pi}))\}$$

$$\Rightarrow (v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E^b_p(\tilde{\pi}))\}$$

$$\geq (v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p(\tilde{\pi}))\}$$

Since, $(v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p(\tilde{\pi}))\} = (1 - \delta)(v - M) > 0$, we have

$$(v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E^b_p(\tilde{\pi}))\} > 0$$

There are two possibilities. Either $p_1(\pi) < p_2(\pi)$ or $p_2(\pi) < p_1(\pi)$. If $p_2(\pi) > p_1(\pi)$, then the buyer can profitably deviate by making an offer of $p_1(\pi)$. The payoff from making such an offer is

$$q(\pi)(v - p_1(\pi)) + (1 - q(\pi))\{\gamma(\pi)\delta v_4(\pi) + (1 - \gamma(\pi))\delta v_B(\pi)\}$$

Since $(v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1-\alpha)(v - E_p^b(\tilde{\pi}))\} > 0$, we can infer that this constitutes a profitable deviation by the buyer.

Next, consider the case when $p_2(\pi) < p_1(\pi)$. In this situation, the buyer can profitably deviate by making an offer of $p_2(\pi)$. The payoff from making such an offer is

$$\{q(\pi)\delta\{\alpha(\pi)(v-M)+(1-\alpha(\pi))(v-E_p^b(\tilde{\pi}))\}+(1-q(\pi))\{\gamma(\pi)(v-p_2(\pi))+(1-\gamma(\pi))\delta v_B(\pi)\}$$

Since $v_4(\pi) < (v - p_2(\pi))$, this constitutes a profitable deviation by the buyer.

This concludes the proof of the lemma. ■

There are two immediate conclusions from the above lemma. First, if $p_2(\pi) < p_1(\pi)$, then it can be shown that if δ is high enough, then in equilibrium, no buyer should offer anything less than $p_1(\pi)$. To show this, suppose at least one of the buyers makes an offer which is less than $p_1(\pi)$ and consider the buyer who makes the lowest offer to S_M. Let $\gamma_1(\pi)$ be the probability with which the other buyer, conditional on making offers to S_M, makes an offer which is less than $p_1(\pi)$. The payoff to the buyer by making the lowest offer to S_M is

$$\{q(\pi)\delta\{\alpha(\pi)(v - M) + (1 - \alpha)(v - E_p^b(\tilde{\pi}))\} + (1 - q(\pi))\delta\{v_B(\pi)\}$$

However, if he makes an offer of $p_1(\pi)$ then the payoff is

$$\{q(\pi)(v - p_1(\pi)) + (1 - q(\pi))\{\gamma_1(\pi)(v - p_1(\pi)) + (1 - \gamma_1(\pi))\delta v_B(\pi)\}$$

We know that as $\delta \to 1$, $v_B(\pi) \to v - H$. Since $p_1(\pi) < H$, this implies that for high δ, $\gamma_1(\pi)(v-p_1(\pi))+(1-\gamma_1(\pi))\delta v_B(\pi) > \delta v_B(\pi)$. Hence, for high δ, this constitutes a profitable deviation by the buyer.

Secondly, if $p_1(\pi) < p_2(\pi)$, then only one buyer can make an offer with positive probability that is less than $p_2(\pi)$. This is because, any buyer who makes an offer to S_M in the range $(p_1(\pi), p_2(\pi))$ can get the offer accepted when the seller S_M gets only one offer. In that case the offer can still get accepted if it is lowered and that will not alter the outcomes following the rejection of the offer. Hence, the buyer can profitably deviate by making a lower offer. Thus, in equilibrium if a buyer has to offer anything less than $p_2(\pi)$ to the seller S_M, then it has to be equal to $p_1(\pi)$. However, in equilibrium both buyers cannot put mass points at $p_1(\pi)$. This shows that only one buyer can make an offer to S_M which is strictly less than $p_2(\pi)$.

Hence, we have argued that all offers to S_M are always greater than or equal to $p_1(\pi)$ and in the event S_M gets two offers, both offers are never below $p_2(\pi)$. This shows that S_M always accepts an equilibrium offer immediately.

We will now show that for any equilibrium in the set of equilibria considered, the informed seller by rejecting equilibrium offers for a finite number of periods can take the posterior to 0. This is shown in the following lemma.

Lemma 3 *Suppose we restrict ourselves to the set of equilibria considered. Given a π and δ, there exists a $T_\pi(\delta) > 0$ such that conditional on getting offers, the informed seller can get an offer of H in $T_\pi(\delta)$ periods from now by rejecting all offers she gets in between. $T_\pi(\delta)$ depends on the sequence of equilibrium offers and corresponding strategies of the responders in the candidate equilibrium. $T_\pi(\delta)$ is uniformly bounded above as $\delta \to 1$.*

Proof. To prove the first part of the lemma, we show that in the candidate equilibrium, rejection of offers by the informed seller can never lead to an upward revision of the belief[14]. If it does, then it implies that the offer is such that the H-type S_I accepts the offer with a positive probability and the L-type S_I rejects it with a positive probability. Since the H-type accepts the offer with a positive probability, this means that the offer must be greater than or equal to H (let this offer be equal to $p_h \geq H$) and we have

$$p_h - H \geq \delta(E' - H)$$

where E' is the price corresponding to the expected equilibrium payoff to the H-type S_I next period. Then,

$$p_h \geq \delta(E') + (1-\delta)H \Rightarrow p_h - L \geq \delta(E') + (1-\delta)H - L$$

$$\Rightarrow p_h - L \geq \delta(E' - L) + (1-\delta)(H - L) > \delta(E' - L)$$

This shows that the L- type S_I should accept p_h with probability[15] 1 . This is a contradiction to our supposition that the L-type S_I rejects with some positive probability. Thus, the belief revision following a rejection must be in the downward direction. It cannot be zero since in that case it implies that both types reject with probability 1. This is not possible in equilibrium.

Thus, in equilibrium, S_I should always accept an offer with a positive probability. This proves the first part of the lemma.

[14]We consider updating in equilibrium. Since this is about a candidate equilibrium, out of equilibrium events could only arise from non-equilibrium offers made by the buyers. However, if we were to follow the definition of the PBE, then no player's action should be treated as containing information about things which that player does not know (*no-signalling-what-you-don't-know*). Hence, these out of equilibrium events cannot lead to change in beliefs.

[15]This follows from the fact that from next period onwards, H-type S_I can always adopt the optimal strategy of the L-type S_I. Hence, following a rejection of the offer p_h, the expected equilibrium payoff to the L-type S_I is $\leq E' - L$

To show that the number of rejections required to get an offer of H is uniformly bounded above as $\delta \to 1$, we need to show that it cannot happen that the acceptance probabilities of any sequence of equilibrium offers to S_I are not uniformly bounded below as $\delta \to 1$.

In the equilibrium considered, if only one buyer makes offers to S_I, then the claim of the lemma holds. This is because of the fact that S_M always accepts an equilibrium offer immediately and hence, S_I on rejecting an offer knows that the continuation game will be a two-player game with one-sided asymmetric information. Thus, by invoking the finiteness result of the two-player game with one-sided asymmetric information, we know that S_I can take the posterior to 0 by rejecting equilibrium offers for finite number of periods.

Consider equilibria where more than one buyer makes offers to S_I. Given the set of equilibria we have considered and the results already proved, we can posit that in such a case, either one of the buyers is making offers only to S_I and the other is randomising between making offers to S_I and S_M, or both buyers are offering to both sellers with positive probabilities.

Let p_l be the minimum offer which gets accepted by S_I with positive probability in an equilibrium where two buyers offer to S_I with positive probability. We will now argue that there exists a possible outcome such that S_I gets only one offer and the offer is equal to p_l. When one of the buyers is making offers to S_I only, then p_l must be the lower bound of the support of his offers . In the second case, when both buyers with positive probability make offers to S_I and S_M, with positive probability S_I gets only one offer. Thus, there exists an instance that S_I gets the offer of p_l only.

When S_I gets the offer of p_l only, then she knows that by rejecting that she gets back a two- player game, which has the finiteness property. Thus there exists a $\tilde{T}(\delta) > 0$ such that S_I is indifferent between getting p_l now and H in $\tilde{T}(\delta)$ periods from now. This implies

$$p_l - L = \delta^{\tilde{T}(\delta)}(H - L)$$

From the finiteness property of the two player game with one sided asymmetric information, we know that $\tilde{T}(\delta)$ is uniformly bounded above as $\delta \to 1$.

Suppose there is a sequence of equilibrium offers such that the acceptance probabilities of the offers are not bounded below as $\delta \to 1$. Let p be the initial offer of that particular sequence. $p \geq p_l$. For a given δ, let $T(\delta) > 0$ be such that, given the acceptance probabilities of the sequence of offers, by rejecting p and subsequent equilibrium offers, S_I can get H in $T(\delta)$ periods from now. Hence, the L-type S_I should be indifferent between getting p now and H in $T(\delta)$ time periods from now. As per our supposition, $T(\delta)$ is not uniformly bounded above as $\delta \to 1$.

Then, we can find a $\delta^h < 1$ such that for all $\delta \in (\delta^h, 1)$, we have $T(\delta) > \tilde{T}(\delta)$. This gives us

$$\delta^{T(\delta)}(H - L) < \delta^{\tilde{T}(\delta)}(H - L) = p_l - L \leq p - L$$

Hence, the L-type S_I is not indifferent between getting p now and H in $T(\delta)$ time periods from now, contrary to our assumption.

Hence, as $\delta \to 1$, probabilities of acceptance of any sequence of equilibrium offers are bounded below. This concludes the proof of the lemma. ■

The above lemma shows that any stationary equilibrium in the set of equilibria considered possess the finiteness property. We will now show that we cannot have both buyers offering to both sellers with positive probability. This is argued in the following lemma.

Lemma 4 *In any equilibrium belonging to the set of equilibria considered, if players are patient enough then both buyers cannot make offers to both sellers with positive probability.*

Proof. From the arguments of lemma (3), we know that in an arbitrary stationary equilibrium, any offer made to the informed seller should get accepted by the low type with a positive probability bounded away from 0. Suppose there exists a stationary equilibrium of the four-player game where both buyers offer to both sellers with a positive probability. Hence, in equilibrium, if the informed seller gets offer(s), then she either gets two offers or one offer. Since S_M always accepts an offer in equilibrium immediately, S_I knows that on rejecting an offer(s) she will get another offer in at most two periods from now. Hence, from lemma (3) we infer that if the informed seller gets one offer, then the L-type S_I can expect to get an offer of H in at most $T_1(\pi) > 0$ time periods from now, by rejecting all offers she gets in between. Similarly, if the informed seller gets two offers then the L-type S_I by rejecting both offers can expect to get an offer of H in at most $T_2(\pi) > 0$ time periods from now by rejecting all offers she gets in between. As we have argued in lemma (3), both $T_1(\pi)$ and $T_2(\pi)$ are bounded above as $\delta \to 1$. Thus, any offer s to the informed seller in equilibrium should satisfy

$$s \geq \delta^{T_1(\pi)}H + (1 - \delta^{T_1(\pi)})L \equiv s_1(\delta)$$

and

$$s \geq \delta^{T_2(\pi)}H + (1 - \delta^{T_2(\pi)})L \equiv s_2(\delta)$$

It is clear from the above that as $\delta \to 1$, both $s_1(\delta) \to H$ and $s_2(\delta) \to H$. Hence, if there is a support of offers to S_I in equilibrium, then the support should collapse as $\delta \to 1$.

We will now argue that for δ high enough but $\delta < 1$, the support in equilibrium cannot have two or more points.

Suppose it is possible that the support of offers to S_I has two or more points. This implies that the upper bound and the lower bound of the support are different from each other. Let $\underline{s}(\pi)$ and $\bar{s}(\pi)$ be the lower and upper bound of the support respectively.

Consider a buyer who is making an offer to S_I. This buyer must be indifferent between making an offer of $\underline{s}(\pi)$ and $\bar{s}(\pi)$. Let $q(\pi)$ be the probability with which the other buyer makes an offer to S_I. Since in equilibrium S_M always accepts an offer immediately, the payoff from making an offer of $\underline{s}(\pi)$ to S_I is

$$\begin{aligned}\Pi_{\underline{s}(\pi)} &= (1-q(\pi))[\alpha_{\underline{s}(\pi)}(v-\underline{s}(\pi)) + (1-\alpha_{\underline{s}(\pi)})\delta v_B(\pi^{'})] \\ &\quad + q(\pi)E_s\{[\beta_\pi^s\delta(v-M) + (1-\beta_\pi^s)\delta v_4(\pi_s^{''})]\}\end{aligned}$$

$\alpha_{\underline{s}(\pi)}$ is the acceptance probability of $\underline{s}(\pi)$ when S_I gets the offer of $\underline{s}(\pi)$ only. β_π^s is the acceptance probability of the offer s to S_I when she gets two offers. $v_B(.)$ and $v_4(.)$ are the buyer's payoffs from the two-player incomplete information game and the four player incomplete information game respectively. For the second term of the right-hand side, we have taken an expectation because when two offers are made, this buyer's offer of $\underline{s}(\pi)$ to S_I never gets accepted and the payoff then depends on the offer made by the other buyer. When S_I gets only one offer and rejects an offer of $\underline{s}(\pi)$, then the updated belief is $\pi^{'}$; $\pi_s^{''}$ denotes the updated belief when S_I rejects an offer of $s \in (\underline{s}(\pi), \bar{s}(\pi)]$ and she gets two offers.

Similarly, the payoff from offering $\bar{s}(\pi)$ is

$$\Pi_{\bar{s}(\pi)} = (1-q(\pi))[\alpha_{\bar{s}(\pi)}(v-\bar{s}(\pi)) + (1-\alpha_{\bar{s}(\pi)})\delta v_B(\pi^{'''})] + q(\pi)[\beta_{2\pi}(v-\bar{s}(\pi)) + (1-\beta_{2\pi})\delta v_4(\pi^4)]$$

Here $\pi^{'''}$ is the updated belief when S_I gets one offer and rejects an offer of $\bar{s}(\pi)$. When S_I gets two offers and rejects an offer of $\bar{s}(\pi)$, the updated belief is denoted by π^4. Note that if at all S_I accepts an offer, she always accepts the offer of $\bar{s}(\pi)$, if made. The quantity $\alpha_{\bar{s}(\pi)}$ is the probability with which the offer of $\bar{s}(\pi)$ is accepted by S_I when she gets one offer. When S_I gets two offers, then the offer of $\bar{s}(\pi)$ gets accepted with probability $\beta_{2\pi}$.

As argued above, $\bar{s}(\pi) \to H$ and $\underline{s}(\pi) \to H$ as $\delta \to 1$. This implies that $v_4(\pi) \to (v-H)$ as $\delta \to 1$. From the result of the two player one-sided asymmetric information game, we know that $v_B(\pi) \to H$ as $\delta \to 1$. Since $v-M > v-H$, we have $\Pi_{\underline{s}(\pi)} > \Pi_{\bar{s}(\pi)}$ as $\delta \to 1$. From lemma (3) we can infer that both β_π^s and $\beta_{2\pi}$ are positive. Hence, there exists a threshold for δ such that if δ crosses that threshold, $\Pi_{\underline{s}(\pi)} > \Pi_{\bar{s}(\pi)}$. This is not possible in equilibrium. Thus, for high δ, the support of offers can have only one point. The same arguments hold for the other buyer as well. Hence, each buyer while offering to S_I has a one-point support. Next, we establish that both buyers should make the same offer. If they make different offers, then as explained before, for δ high enough the buyer making the higher offer can profitably deviate by making the lower offer. However, in equilibrium it is not possible to have both buyers making the same offer to S_I[16]

Hence, when agents are patient enough, in equilibrium both buyers cannot offer to both sellers with a positive probability. This concludes the proof of the lemma. ■

In the following lemma we show that in any stationary equilibrium of the four player game, as players get patient enough, S_M always gets offers from two buyers with a positive probability.

Lemma 5 *In any stationary equilibrium belonging to the set of equilibria considered, there exists a threshold of δ such that if δ exceeds that threshold, both buyers make offers to S_M with positive probability.*

Proof. Suppose there exists a stationary equilibrium where S_M gets offers from only one buyer, say B_1. First, we argue that in such a stationary equilibrium, if the buyer offering to S_M offers something greater than or equal to M, then S_M accepts it immediately. To explain this, let $p_m \geq M$ be the offer made by the buyer who makes offers to S_M. Then, S_M on rejecting this offer either gets back a two-player game or a four-player game. In either case, she cannot expect to get anything more than p_m. Hence, she immediately accepts it. This implies that if there is a stationary equilibrium where S_M gets offers from only one buyer then that buyer should always offer M to S_M and S_M immediately accepts it.

There can, therefore, be two possibilities. Either S_I gets an offer from B_2 only or from both B_1 and B_2 with positive probability. Consider the first case. Since S_M will accept the offer immediately, B_2, must be making an offer greater than or equal to p^e, such that

$$p^e = (1-\delta)L + \delta(H-\epsilon)$$

where $\epsilon > 0$ and $\epsilon \to 0$ as $\delta \to 1$. This is because in equilibrium, if S_I rejects an offer then next period she faces a two-player game. This game has a unique equilibrium and the price offers in that equilibrium goes to H as $\delta \to 1$.

From this we can infer that there exists a threshold of δ such that if δ exceeds that threshold then $p^e > M$.

Hence, B_2 can profitably deviate, contradicting the hypothesis of equilibrium.

In the latter case, we know that B_1 offers to both S_I and S_M with positive probability and B_2 makes offers only to S_I. Therefore, using the result of lemma (3), if B_1 has to get an

[16]These arguments would also work even if the supports were not taken to be symmetric. In that case, let $\underline{s}(\pi)$ be the minimum of the lower bounds and $\bar{s}(\pi)$ be the maximum of the upper bounds. If these are associated with the same buyer, then same arguments hold. If not, then the buyer with the higher upper bound can proftibaly deviate by shifting its mass to $\underline{s}(\pi)$.

offer accepted by S_I, then for high values of δ that offer should be close to H and thus the payoff to B_1 from making offers to S_I should be close to $(v - H)$. On the other hand, the payoff to B_1 from making offers to S_M is $(v - M)$. However, in equilibrium, the buyer has to be indifferent between making offers to S_I and S_M. Hence, it is not possible to have a stationary equilibrium where S_M gets offers from only one buyer. This concludes the proof. ■

From the characteristics of the restricted set of equilibria being considered, we know that S_M always gets an offer with a positive probability. The above lemma then allows us to infer that, in any stationary equilibrium of the four player game, both buyers should offer to S_M with positive probability. From our arugments and hypothesis, we know that both buyers cannot make offers to only one seller (S_I or S_M) and both buyers cannot randomise between making offers to both sellers. Hence, we can infer that one of the buyers has to make offers to S_M only and the other buyer should randomise between making offers to S_I and S_M.

The following lemma now shows that for any $\pi \in [0, 1)$, any equilibrium in this restricted set possesses the characteristic that the price offers to all sellers approach H as $\delta \to 1$.

Lemma 6 *For a given π, in any hypothesised equilibrium, price offers to all sellers go to H as $\delta \to 1$.*

Proof. Let $\bar{s}(\pi)$ be the upper bound of the support[17] of offers to S_M.

S_M always accepts an equilibrium offer immediately. Hence, if the L-type S_I rejects an equilibrium offer, she gets back a two-player game with one-sided asymmetric information. Thus, the buyer offering to S_I in a period must offer at least p^e such that

$$p^e - L = \delta(H - \epsilon - L) \Rightarrow p^e = (1 - \delta)L + \delta(H - \epsilon)$$

where $\epsilon > 0$ and $\epsilon \to 0$ as $\delta \to 1$.

Consider B_1, who is randomising between making offers to S_I and S_M. When offering to S_I, B_1 must offer p^e and it must be the case that

$$(v - p^e)\alpha(\pi) + (1 - \alpha(\pi))\delta\{v - (H - \epsilon)\} = v - \bar{s}(\pi)$$

where $\alpha(\pi)$ is the probability with which the offer is accepted by the informed seller. This follows from the fact that B_1 must be indifferent between offering to S_I and S_M. The L.H.S of the above equality is the payoff to B_1 from offering to S_I and the R.H.S is the payoff to him from offering to S_M. Since in any hypothesized equilibrium, S_M always gets an offer in

[17] The upper bound of support of offers to S_M for both buyers should be the same. Else, the buyer with the higher upper bound can profitably deviate

period 1 and S_M accepts an equilibrium offer immediately, S_I, by rejecting an equilibrium offer, always gets back a two-player game with one-sided asymmetric information. Hence, the payoff to the buyer from offering to S_I is the same as in the two-player game with one-sided asymmetric information. This implies that

$$(v - p^e)\alpha(\pi) + (1 - \alpha(\pi))\delta\{v - (H - \epsilon)\} = v_B(\pi)$$

Thus, we can conclude that $v_B(\pi) = v - \bar{s}(\pi)$.

We will now show that the upper bound of the support of offers to S_M is strictly greater than p^e. We have

$$(v - p^e) - \delta\{v - (H - \epsilon)\} = v(1 - \delta) + \delta(H - \epsilon) - \delta(H - \epsilon) - (1 - \delta)L$$

$$= (1 - \delta)(v - L) > 0$$

for $\delta < 1$. This implies that

$$v - p^e > \delta\{v - (H - \epsilon)\}$$

Since $(v - \bar{s}(\pi))$ is a convex combination of $v - p^e$ and $\delta\{v - (H - \epsilon)\}$, we have

$$v - p^e > v - \bar{s}(\pi) \Rightarrow \bar{s}(\pi) > p^e$$

Next, we will argue that as $\delta \to 1$, the support of offers to S_M from any buyer is bounded below by p^e. Consider a buyer who makes an offer of p^e to S_M in equilibrium. Then, if q^e is the probability with which this offer gets accepted, we have

$$(v - p^e)q^e + (1 - q^e)\delta v_B(\pi) = v - \bar{s}(\pi)$$

This follows since S_M always accepts an offer in equilibrium immediately, this buyer's offer to S_M gets rejected only when the other buyer also makes an offer to S_M.

This gives us,

$$q^e = \frac{(v - \bar{s}(\pi)) - \delta v_B(\pi)}{(v - p^e) - \delta v_B(\pi)} = \frac{(1 - \delta)(v - \bar{s}(\pi))}{(v - p^e) - \delta(v - \bar{s}(\pi))}$$

$$\Rightarrow q^e = \frac{1}{\frac{v}{v - \bar{s}(\pi)} + \frac{\delta\bar{s}(\pi) - p^e}{(1-\delta)(v - \bar{s}(\pi))}}$$

and

$$q^e \to 0 \text{ as } \delta \to 1$$

This shows that in equilibrium, as $\delta \to 1$, any offer to S_M that is less than or equal to p^e always gets rejected. Since we have argued earlier that in equilibrium, no buyer should make an offer to S_M that she always rejects, we can infer that the support of offers to S_M from any buyer is bounded below by p^e as δ approaches 1. Hence, in any arbitrary stationary equilibrium of this kind, the price offers to all sellers are bounded below by p^e as δ approaches 1. However, as $\delta \to 1$, $p^e \to H$. Hence, as $\delta \to 1$, the support of offers to S_M from any buyer collapses and hence price offers to all sellers converge to H. ■

Thus, we have shown that for any stationary equilibrium in the set of equilibria considered, one of the buyers randomises between making offers to S_M and S_I and the other buyer makes offers to S_M only. Further, as $\delta \to 1$, price offers in all transactions in these stationary equilibria go to H. This concludes the proof of the proposition. ■

We will now argue that there does not exist any stationary equilibrium where both buyers offer only to S_I. This is done in the following lemma.

Lemma 7 *Let Π be the set of beliefs such that for $\pi \in \Pi$, it is possible to have a stationary equilibrium where both buyers offer only to S_I. The set Π is empty*

Proof. We begin the proof by first showing that the set Π^c is non-empty. Suppose not. Then, for all π, it is possible to have a stationary equilibrium where both buyers make offers only to S_I. In this case, price offers can never exceed M. This is because S_M does not get any offer in the presence of all four players. Thus, if any buyer unilaterally deviates and offers a price greater than or equal to M to S_M, S_M will accept it.

Let $\bar{p}$ be the largest price offer, for any π, in such an equilibrium. (Clearly, such an offer exists.) This offer is accepted by the L -type S_I with probability 1, since the payoff from rejecting can be at most $\delta(\bar{p} - L)$ and $\delta < 1$. But then, in the following period, $\pi = 0$ (by Bayes' Theorem) and, therefore, as $\delta \to 1$, the payoff to S_I from such a continuation game is close to $H - L$ ($> M - L$). Since $\bar{p} \leq M$, the L-type S_I can unilaterally deviate to get a higher payoff and hence, this cannot be an equilibrium. This shows that Π^c is non-empty.

Suppose now that Π is non-empty. Consider any $\pi \in \Pi$. As explained earlier, no equilibrium can involve offers that are rejected by both L and H types. Therefore, the L type must accept an offer with positive probability. This implies (by Bayes' Theorem and $\delta < 1$) that the sequence of prices must be increasing. Also, by hypothesis, the price is bounded above by M. Let $\bar{p}'$ be the largest price offer in such an equilibrium. As argued before, $\bar{p}' \leq M$. There are two possibilities. Either the updated belief conditional on $\bar{p}'$ being rejected is in Π or it is in Π^c. In the former case, S_I should accept the offer with probability 1 and the updated belief is $\pi = 0$, where the equilibrium price offer must be $H > M$, leading to the existence of a profitable deviation, for δ sufficiently high. For the latter case, if δ is high then from proposition (1) we know that for any stationary equilibrium all offers converge to H as $\delta \to 1$. Once again, this implies the existence of a profitable deviation for the L-type S_I. Hence, we cannot have Π non-empty. This concludes the proof. ■

We now state our main result of the paper in the theorem below

Theorem 1 *In any arbitrary stationary equilibrium of the four-player game, as the discount factor goes to 1, price offers in all transactions converge to H for all values of the prior $\pi \in [0, 1)$.*

Proof. The proof the theorem follows directly from proposition (1) and lemma (7). ■

The following subsection now constructs one stationary equilibrium of the four player game with incomplete information.

3.4 Characterisation of an equilibrium of the four-player game with incomplete information.

In this subsection, we will characterise a particular stationary equilibrium of the incomplete information game with four players. The analysis so far has established that any stationary equilibrium of the game should have certain qualitative features. Here, we assume that $L = 0$, for purposes of reducing notation. The main result of this subsection is described in the following proposition.

Proposition 2 *There exists a $\delta^* \in (0, 1)$ such that if $\delta > \delta^*$, then for all $\pi \in [0, 1)$ there exists a stationary equilibrium as follows (both public and private offers:):*

(i) One of the buyers (say B_1) will make offers to both S_I and S_M with positive probability. The other buyer B_2 will make offers to S_M only.

(ii) B_2 while making offers to S_M will put a mass point at $p_l^{'}(\pi)$ and will have an absolutely continuous distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$ where $p_l^{'}(\pi)$ $(p_l(\pi))$ is the minimum acceptable price to S_M when she gets one(two) offer(s). For a given π, $\bar{p}(\pi)$ is the upper bound of the price offer S_M can get in the described equilibrium $(p_l^{'}(\pi) < p_l(\pi) < \bar{p}(\pi))$. B_1 while making offers to S_M will have an absolutely continuous (conditional) distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$, putting a mass point at $p_l(\pi)$.

(iii) B_1 while making offers to S_I on the equilibrium path behaves exactly in the same manner as in the two player game with one-sided asymmetric information.

(iv) S_I's behavior is identical to that in the two-player game. S_M accepts the largest offer with a payoff at least as large as the expected continuation payoff from rejecting all offers.

(v) Each buyer in equilibrium obtains a payoff of $v_B(\pi)$.

Remark 1 *The mass points and the distribution of buyers' offers will depend upon π though we show that these distributions will collapse in the limit. Off the path, the analysis is different from the two-player game because the buyers have more options to consider when choosing actions. For the description of off-path behavior refer to Appendix(B).*

Remark 2 *A "road map" of the proof: We construct the equilibrium by starting from the benchmark complete information case and showing that the complete information strategies essentially carry over to the game where π is in a range near 0. This includes, through the competition lemma, showing the nature of the competition among the sellers. Once π is outside this range, the mass points and support of the randomised strategies in the candidate equilibrium will depend upon π and these are characterised for all values of π. The equilibrium is then extended beyond the initial range (apart from the initial range, these are functions of δ) for sufficiently high values of δ by recursion. Finally, checking that the candidate equilibrium is immune to unilateral deviation at any stage involves specifying out-of-equilibrium beliefs. This is done in the appendix.*

Proof. We prove this proposition in steps. (Not all of these steps are given here in order to reduce unwieldy notation-see also the appendix.) First we derive the equilibrium for a given value of π by assuming that there exists a threshold δ^*, such that if δ exceeds this threshold then for each value of π, a stationary equilibrium as described above exists. Later on we will prove this existence result.

To formally construct the equilibrium for different values of π, we need the following lemma which we label as the *competition lemma*, following the terminology of [9], though they proved it for a different model.

Consider the following sequences for $t \geq 1$:

$$\bar{p}_t = v - [(v - \delta^t H)\alpha + (1 - \alpha)\delta(v - \bar{p}_{t-1})] \tag{2}$$

$$p_t^{'} = M + \delta(1 - \alpha)(\bar{p}_{t-1} - M) \tag{3}$$

where $\alpha \in (0, 1)$ and $\bar{p}_0 = H$.

Lemma 8 *There exists a* $\delta' \in (0,1)$*, such that for* $\delta > \delta'$ *and for all* $t \in \{1, ...N(\delta)\}$*, we have,*

$$\bar{p}_t > p_t'$$

Proof.

$$\begin{aligned}\bar{p}_t - p_t' &= v - [(v - \delta^t H)\alpha + (1-\alpha)\delta(v - \bar{p}_{t-1}))] - M \\ &\quad -\delta(1-\alpha)(\bar{p}_{t-1} - M)\end{aligned}$$

$$= (v - M)(1 - \delta + \delta\alpha) - \alpha(v - \delta^t H)$$

$$= (1-\delta)(v - M) + \alpha(\delta v - \delta M - v + \delta^t H)$$

$$= (1-\delta)(v - M) + \alpha(\delta^t H - \delta M - (1-\delta)v)$$

If we show that the second term is always positive then we are done. Note that the coefficient of α is increasing in delta and is positive at $\delta = 1$. Take $t = N^*$, where N^* is the upper bound on the number of periods up to which the two player game with one sided asymmetric information (as described earlier) can continue. For $t = N^*$, $\exists\ \delta' < 1$ such that the term is positive whenever $\delta > \delta'$. Since this is true for $t = N^*$, it will be true for all lower values of t.

As $N(\delta) \leq N^*$ for any $\delta < 1$ and for all $t \in \{1, ...N(\delta)\}$,

$$\bar{p}_t > p_t'$$

whenever $\delta > \delta'$.

This concludes the proof of the lemma. ■

Fix a $\delta > \delta^*$. Suppose we are given a $\pi \in (0,1)$[18]. There exists a $t \geq 0$ (it is easy to see that this $t \leq N^*$) such that $\pi \in [d_t, d_{t+1})$. The sequence $d_\tau(\delta) = \{0, d_1, d_2, ...d_t..\}$ is derived from and is identical with the same sequence in the two-player game. Next, we evaluate $v_B(\pi)$ (from the two player game). Define $\bar{p}(\pi)$ as,

$$\bar{p}(\pi) = v - v_B(\pi)$$

Define $p_l'(\pi)$ as,

$$p_l'(\pi) = M + \delta(1 - a(\pi))[E_{d_{t-1}}(p) - M] \tag{4}$$

where $E_{d_{t-1}}(p)$ represents the expected price offer to S_M in equilibrium when the probability that S_I is of the low type is d_{t-1}. From (4) we can posit that, in equilibrium, $p_l'(\pi)$ is the

[18] $\pi = 0$ is the complete information case with a H seller.

minimum acceptable price for S_M if she gets only one offer.

Lemma 9 *For a given $\pi > d_1$, the acceptance probability $a(\pi, \delta)$ of an equilibrium offer is increasing in δ and has a limit $\bar{a}(\pi)$ which is less than* 1*.*

Proof. The acceptance probability $a(\pi, \delta)$ of an equilibrium offer is equal to $\pi\beta(\pi, \delta)$, where $\beta(\pi, \delta)$ is the probability with which the L-type S_I accepts an equilibrium offer. From the updating rule we know that $\beta(\pi, \delta)$ is such that the following relation is satisfied:

$$\frac{\pi(1 - \beta(\pi, \delta))}{\pi(1 - \beta(\pi, \delta)) + (1 - \pi)} = d_{t-1}(\delta)$$

From the above expression, we get

$$\beta(\pi, \delta) = \frac{\pi - d_{t-1}(\delta)}{\pi(1 - d_{t-1}(\delta))}$$

Since $d_t's$ have a limit as δ goes to 1, so does $\beta(\pi, \delta)$. Therefore, $a(\pi, \delta)$ also has a limit $\bar{a}(\pi)$ which is less than 1 for $\pi \in (0, 1)$. ■

For $\pi = d_{t-1}$, the maximum price offer to S_M (according to the conjectured equilibrium) is $\bar{p}(d_{t-1})$. This implies that $E_{d_{t-1}}(p) < \bar{p}(d_{t-1})$ (this will be clear from the description below). Since $a(\pi) \in (0, 1)$, from lemma (8) we can infer that $\bar{p}(\pi) > p_l^{'}(\pi)$. Suppose there exists a $p_l(\pi) \in (p_l^{'}(\pi), \bar{p}(\pi))$ such that,

$$p_l(\pi) = (1 - \delta)M + \delta E_\pi(p)$$

We can see that p_l represents the minimum acceptable price offer for S_M in the event that he gets two offers. (Note that if S_M rejects both offers, the game goes to the next period with π remaining the same.)

From the conjectured equilibrium behavior, we derive the following[19] :

1. B_1 makes offers to S_I with probability $q(\pi)$, where

$$q(\pi) = \frac{v_B(\pi)(1 - \delta)}{(v - p_l^{'}(\pi)) - \delta v_B(\pi)} \tag{5}$$

B_1 offers $\delta^t H$ to S_I. With probability $(1 - q(\pi))$ he makes offers to S_M. The conditional distribution of offers to S_M, given B_1 makes an offer to this seller when the relevant probability

[19] We obtain these by using the indifference relations of the players when they are using randomized behavioral strategies.

is π, is

$$F_1^\pi(s) = \frac{v_B(\pi)[1-\delta(1-q(\pi))] - q(\pi)(v-s)}{(1-q(\pi))[v-s-\delta v_B(\pi)]} \tag{6}$$

We can check that $F_1^\pi(p_l(\pi)) > 0$ and $F_1^\pi(\bar{p}(\pi)) = 1$. This confirms that B_1 puts a mass point at $p_l(\pi)$.

2. B_2 offers $p_l^{'}(\pi)$ to S_M with probability $q^{'}(\pi)$, where

$$q^{'}(\pi) = \frac{v_B(\pi)(1-\delta)}{(v-p_l(\pi))) - \delta v_B(\pi)} \tag{7}$$

With probability $(1-q^{'}(\pi))$ he makes offers to S_M by randomizing his offers in the support $[p_l(\pi), \bar{p}(\pi)]$. The conditional distribution of offers is given by

$$F_2^\pi(s) = \frac{v_B(\pi)[1-\delta(1-q^{'}(\pi))] - q^{'}(\pi)(v-s)}{(1-q^{'}(\pi))[v-s-\delta v_B(\pi)]} \tag{8}$$

This completes the derivation. Appendix(B) describes the off-path play and show that it sustains the equilibrium play

Next, we show that there exists a δ^* such that $\delta^{'} < \delta^* < 1$ and for $\delta > \delta^*$ an equilibrium as described above exists for all values of $\pi \in [0,1)$. To do these we need the following lemmas:

Lemma 10 *If $\pi \in [0, d_1)$, then the equilibrium of the game is identical to that of the benchmark case.*

Proof. From the equilibrium of the two player game with one sided asymmetric information, we know that for $\pi \in [0, d_1)$, buyer always offers H to the seller and the seller accepts this with probability one. Hence this game is identical to the game between a buyer of valuation v and a seller of valuation H, with the buyer making the offers. Thus, in the four-player game, we will have an equilibrium identical to the one described in the benchmark case. We conclude the proof by assigning the following values:

$$p_l^{'}(\pi) = M \text{ and } \bar{p}(\pi) = H \text{ for } \pi \in [0, d_1)$$

■

Lemma 11 *If there exists a $\bar{\delta} \in (\delta^{'}, 1)$ such that for $\delta \geq \bar{\delta}$ and for all $t < T$ $(T < N^*\}$ an equilibrium exists for $\pi \in [0, d_t(\delta))$, then there exists a $\delta_T^* \geq \bar{\delta}$ such that, for all $\delta \in (\delta_T^*, 1)$ an equilibrium also exists for $\pi \in [d_T(\delta), d_{T+1}(\delta))$.*

Proof. We only need to show that there exists a $\delta^*_T \geq \bar{\delta}$ such that for all $\delta > \delta^*_T$ and for all $\pi \in [d_T(\delta), d_{+1}(\delta))$,there exists a $p_l(\pi) \in (p_l^{'}(\pi), \bar{p}(\pi))$ with

$$p_l(\pi) = (1 - \delta)M + \delta E_\pi(p)$$

From now on we will write d_T instead of $d_T(\delta)$. For each $\delta \in (\delta^{'}, 1)$ we can construct $d(\delta)$ and the equilibrium strategies as above (assuming existence). For any $x \in (p_l^{'}(\pi), \bar{p}(\pi))$, construct the function $G(x)$ as

$$G(x) = x - [\delta E^x_\pi(p) + (1 - \delta)M]$$

We can infer from ([7]) that the function $G(.)$ is monotonically increasing in x. Since $E^x_\pi(p) < \bar{p}(\pi)$,

$$\lim_{x \to \bar{p}(\pi)} G(x) > 0$$

Next, we have

$$G(p_l^{'}(\pi)) = p_l^{'}(\pi) - [\delta E_\pi^{p_l^{'}(\pi)}(p) + (1 - \delta)M]$$

By definition $E_\pi^{p_l^{'}(\pi)}(p) > p_l^{'}(\pi)$. So for $\delta = 1$, $G(p_l^{'}(\pi))) < 0$. Since $G(.)$ is a continuous function, there exists a $\delta^*_T \geq \bar{\delta}$ such that for all $\delta > \delta^*_T$, $G(p_l^{'}(\pi))) < 0$. By invoking the *Intermediate Value Theorem* we can say that there is a unique $x^* \in (p_l^{'}(\pi), \bar{p}(\pi))$ such that $G(x^*) = 0$. This x^* is our required $p_l(\pi)$.

This concludes the proof. ■

From lemma (10) we know that for any $\delta \in (0, 1)$ an equilibrium exists for $\pi \in [0, d_1)$.[20] Using lemma (11) we can obtain δ^*_t for all $t \in \{1, 2, ..., N^*\}$. Define δ^* as:

$$\delta^* = \max_{1,..,N^*} \delta^*_T$$

We can do this because N^* is finite. Lemma (10) and (11) now guarantee that whenever $\delta > \delta^*$ an equilibrium as described above exists for all $\pi \in [0, 1)$.

This concludes the proof of the proposition. ■

4 Asymptotic characterization

In this section we show that in the equilibrium characterised, as $\delta \to 1$, price offers in all transactions go to H. This is the unique asymptotic outcome of any stationary equilibrium

[20]Note that d_1 is independent of δ

of the game, as described earlier in the paper.

It has been argued earlier that as $\delta \to 1$, $p_l^{'}(\pi)$ reaches a limit which is less than $\bar{p}(\pi)$. From (5) we then have,

$$q(\pi) \to 0 \text{ as } \delta \to 1$$

Then from (6) we have,

$$1 - F_1^{\pi}(s) = \frac{\bar{p}(\pi) - s}{(1 - q(\pi))[v - s - \delta v_B(\pi)]}$$

We have shown that $q(\pi) \to 0$ as $\delta \to 1$. Hence as $\delta \to 1$, for s arbitrarily close to $\bar{p}(\pi)$, we have

$$1 - F_1^{\pi}(s) \approx \frac{\bar{p}(\pi) - s}{\bar{p}(\pi) - s} = 1$$

Hence the distribution collapses and $p_l(\pi) \to \bar{p}(\pi)$. From the expression of $p_l(\pi)$ we know that $p_l(\pi) \to E_{\pi}(p)$ as δ goes to 1. Thus we can conclude that $E_{\pi}(p)$ approaches $\bar{p}(\pi)$. From the two-player game with one-sided asymmetric information we know that as δ goes to 1, $\bar{p}(\pi) \to H$, (since $v_B(\pi)$ goes to $v - H$) for any value of π. This leads us to conclude that as δ goes to 1, $E_{\pi}(p) \to H$ for all values of π. This in turn provides the justification of having $E_{d_{t-1}}(p) \approx E_{\pi}^{x}(p)$ for high values of δ(used in the proof of lemma (11)).

From the ([7]) we know that $G(\bar{p}(\pi)) > 0$. Hence there will be a threshold of δ such that for all δ higher than that threshold we have $G(\delta\bar{p}(\pi)) > 0$. Thus $p_l(\pi)$ is bounded above by $\delta\bar{p}(\pi)$. (7) implies that

$$q^{'}(\pi) = \frac{1}{\frac{v}{v_B(\pi)} + \frac{\delta\bar{p}(\pi) - p_l(\pi)}{(1-\delta)v_B(\pi)}}$$

Since $p_l(\pi)$ is bounded above by $\delta\bar{p}(\pi)$, $q^{'}(\pi) \to 0$ as δ goes to 1.

Thus we conclude that as δ goes to 1, prices in all transactions go to H.

Comment: It should be mentioned that we would expect the same result to be true, if, instead of a two-point distribution, the informed type's reservation value s *is continuously distributed in* $(L, H]$ *according to some cdf* $G(s)$. Appendix (H) describes this in detail.

In the following section we discuss some extensions.

5 Extensions

In this section we consider some possible extensions by having offers to be private and by considering a non-stationary equilibrium

5.1 A non-stationary equilibrium

We show that with public offers we can have a non-stationary equilibrium, so that the equilibrium constructed in the previous sections is not unique. This is based on using the stationary equilibrium as a punishment (the essence is similar to the *pooling equilibrium with positive profits* in [30]). The strategies sustaining this are described below. The strategies will constitute an equilibrium for sufficiently high δ, as is also the case for the stationary equilibrium.

Suppose for a given π, both the buyers offer M to S_M. S_M accepts this offer by selecting each seller with probability $\frac{1}{2}$. If any buyer deviates, for example by offering to S_I or making a higher offer to M, then all players revert to the stationary equilibrium strategies described above. If S_M gets the equilibrium offer of M from the buyers and rejects both of them then the buyers make the same offers in the next period and the seller S_M makes the same responses as in the current period.

Given the buyers adhere to their equilibrium strategies, the continuation payoff to S_M from rejecting all offers she gets is zero. So she has no incentive to deviate. Next, if one of the buyers offers slightly higher than M to S_M then it is optimal for her to reject both the offers. This is because on rejection next period players will revert to the stationary equilibrium play described above. Hence her continuation payoff is $\delta(E_\pi(p) - M)$,which is higher than the payoff from accepting.

Finally each buyer obtains an equilibrium payoff of $\frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$. If a buyer deviates then, according to the strategies specified, S_M should reject the higher offer if the payoff from accepting it is strictly less than the continuation payoff from rejecting(which is the one period discounted value of the payoff from stationary equilibrium). Hence if a buyer wants S_M to accept an offer higher than M then his offer $p^{'}$ should satisfy,

$$p^{'} = \delta E_\pi(p) + (1 - \delta)M$$

The payoff of the deviating buyer will then be $\delta(v - E_\pi(p)) + (1 - \delta)(v - M)$. As $\delta \to 1$, $\delta(v - E_\pi(p)) + (1 - \delta)(v - M) \approx \delta(v - \bar{p}(\pi) + (1 - \delta)(v - M)$
$= \delta v_B(\pi) + (1 - \delta)(v - M)$.

For $\delta = 1$ this expression is strictly less than $\frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$, as $(v - M) > \delta v_B(\pi)$. Hence for sufficiently high values of δ this will also be true. Also if a buyer deviates and makes an offer in the range $(M, p^{'})$ then it will be rejected by S_M. The continuation payoff of the buyer will then be $\delta v_B(\pi) < \frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$. Hence we show that neither buyer has any incentive to deviate.

We conclude this section by arguing that it is not possible to establish this equilibrium with private offers. This is because the equilibrium presented in this section depends crucially on "overbidding" by one of the buyers to S_M being detected by the other buyer and seller, who then condition their future play on this deviation. But with private offers, this deviation is not detectable, so the switch to the "punishment phase" is not possible.

5.2 Private offers

In this subsection, we consider a variant of the extensive form of the four-player incomplete information game by in which offers are private to the recipient and the proposer. This means in each period a seller observes only the offer(s) she gets and a buyer does not know what offers are made by the other buyer or received by the sellers.

The equilibrium notion here is that of a *public perfect Bayesian equilibrium (PPBE)*. That is, in equilibrium strategies can condition only on the public history, (which is the set of players remaining in a particular period) and the public belief. Hence, the equilibrium characterised for the 4 -player incomplete information game with public offers is a PPBE of the game with private targeted offers, with off-path behaviour as described in Appendix(C).

In case of public targeted offers, while proving that the stationary equilibrium outcome is unique, we did not use the fact that each seller while responding observes the other seller's offer.

Any stationary equilibrium of the public targeted offers game is a particular public perfect Bayesian equilibrium of the game with private targeted offers. We now argue that no non-stationary equilibrium with public offers can be established as an equilibrium with private offers. Any non-stationary equilibrium depends on the expected future changes due to publicly observed deviations. With private offers, the only deviation which is observed, results in either some subset of players leaving or all players remaining. In the first case, the continuation game will be a 2 player game. These continuation games have unique equilibria. This implies that in any non-stationary equilibrium, there has to be a possible off-path play where all four players remain. However, in that case, the player who deviated needs to be detected for proper specification of continuation play after a deviation. This is not possible with private offers. Hence, the asymptotic outcome implied by the described PPBE of the private targeted offers model is the unique asymptotic PPBE outcome.

6 Conclusion

In the model we described above we have shown that the unique stationary PBE outcome has the property that , as $\delta \to 1$, the prices in both transactions go to the highest seller value H, with the buyers making offers. If one interprets the alternative possibilities on the opposite side of the market as outside options, this result restores the Coase conjecture for bilateral bargaining with one-sided private information in the presence of outside options. With *private offers,* the uniqueness result is for public perfect Bayes' equilibrium, since stationarity does not have any bite in this context. We then construct a stationary equilibrium to show existence.

We have shown that there could be non-stationary equilibria in this model and have also considered extensions to a continuum of seller types. The continuum of types does not affect the result.

In our future research we intend to address the issue of having two privately informed sellers and to extend this model to more agents on both sides of the market.

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Appendix

A Off-path behavior of the 2 player game with incomplete information

We recapitulate here the off-path beliefs that sustain the equilibrium we have discussed for the two-player game. Suppose, for a given δ and π, the equilibrium offer is $\delta^t H$(i.e $\pi \in [d_t, d_{t+1})$) .We need to consider the following off-path contingencies.

(a) The buyer offers p^o to the seller such that $p^o < \delta^t H$: If $p^0 < \delta^{t+1} H$ then both the L-type and H-type seller reject this offer with probability 1. If $p^o \in [\delta^{t+1} H, \delta^t H)$ then the L-type seller rejects this with a probability, which, through Bayes' rule, implies that the updated belief is d_t. Let this probability be $\beta''(p)$. Hence the acceptance probability of this offer is $a''(p) = \pi \beta''(p)$. The H-type seller always rejects this offer. Since $p^o \in [\delta^{t+1} H, \delta^t H)$, there exists a $k \in (0, 1]$ such that $p^o = k\delta^{t+1} H + (1-k)\delta^t H$. Next period (if the seller rejects now) the buyer offers $\delta^t H$ with probability k and $\delta^{t-1} H$ with probability $(1 - k)$. This is optimal from the point of view of the buyer because at $\pi = d_t$, the buyer is indifferent between offering $\delta^t H$ and $\delta^{t-1} H$. Also the expected continuation payoff to the L-type seller from rejection is equal to $\delta(k\delta^t H + (1 - k)\delta^{t-1} H) = p^o$. Thus the L-type seller is indifferent between accepting and rejecting the offer of p^o.

The way the cutoffs d_t's are derived ensures that the buyer has no incentive to deviate and offer something less than $\delta^t H$.

(b) Next, consider the case when the buyer offers p^o to the seller such that $p^o > \delta^t H$. If $p^o \in (\delta^t H, \delta^{t-1} H]$, the L-type seller rejects this offer with a probability that takes the updated belief to d_{t-1}. Since $p^o \in (\delta^t H, \delta^{t-1} H]$, there exists a $k \in (0, 1]$, such that $p^o = k\delta^{t-1}H + (1-k)\delta^t H$. If the seller rejects then next period the buyer offers $\delta^{t-2}H$ with probability k and $\delta^{t-1}H$ with probability $1-k$. This is optimal from the buyer's point of view since at $\pi = d_{t-1}$, the buyer is indifferent between offering $\delta^{t-1}H$ and $\delta^{t-2}H$. Since the expected payoff to the L-type seller from rejection is $\delta(k\delta^{t-2}H + (1-k)\delta^{t-1}H) = p^o$, he is indifferent between accepting and rejecting an offer of p^o. As p^o is strictly greater than $\delta^t H$ and the acceptance probability is the same as that of the equilibrium offer, the buyer has no incentive to deviate and offer p^o to the seller where $p^o \in (\delta^t H, \delta^{t-1} H]$.

If $p^o \in (\delta^\tau, \delta^{\tau-1}]$ (for $\tau \leq t-1$) then the L-type seller rejects this with a probability which through Bayes' rule implies that the updated belief is $d_{\tau-1}$. If the seller rejects then next period the buyer randomises between offering $\delta^{\tau-1}H$ and $\delta^{\tau-2}H$ such that the expected continuation payoff to the L-type seller from rejection is p^o. It can be checked that the buyer has no incentive to deviate and offer p^o where $p^o \in (\delta^\tau, \delta^{\tau-1}]$ ($\tau \leq t-1$).

B Off-path behavior of the 4 player game with incomplete information(public offers)

Suppose B_2 adheres to his equilibrium strategy. Then the off-path behavior of B_1 and that of L-type S_I, while B_1 makes an offer greater than $\delta^t H$ to S_I, are the same as in the 2-player game with incomplete information. If B_1's offer to S_I is less than $\delta^t H$ then the off-path behavior of the L-type S_I is described in the following manner. If B_2's offer to S_M is in the range $[p_l(\pi), \bar{p}(\pi)]$, then the L-type S_I behaves in the same way as in the 2-player game. If B_2 offers $p_l^{'}(\pi)$ to S_M then the L-type S_I accepts the offer with the equilibrium probability so that rejection takes the posterior to d_{t-1}. Next period, B_1 randomises between d_{t-1} and d_{t-2} so that the L-type S_I is indifferent between accepting or rejecting the offer now. For high values of δ, B_1 has no incentive to deviate.

Next, suppose B_2 makes an unacceptable offer to S_M, (which is observable to S_I) and B_1 makes an equilibrium offer to S_I. The L-type S_I rejects this offer with a probability that takes the updated belief to d_{t-1}. If S_I rejects this equilibrium offer and next period both the buyers make offers to S_M, then two periods from now, the remaining buyer offers $\delta^{t-2}H$ (the buyer is indifferent between offering $\delta^{t-1}H$ and $\delta^{t-2}H$ at $\pi = d_{t-1}$) to S_I. Thus the expected continuation payoff to S_I from rejection is $\delta(q(d_{t-1})\delta^{t-1}H + \delta(1 - q(d_{t-1}))\delta^{t-2}H) = \delta^t H$. This implies that the L-type S_I is indifferent between accepting and rejecting an offer of $\delta^t H$ if he observes S_M to get an unacceptable offer.

Now consider the case when B_2 deviates and makes an offer to S_I. It is assumed that if S_I gets two offers then she disregards the lower offer.

Suppose B_1 makes an equilibrium offer to S_I and B_2 deviates and offers something less than $\delta^t H$ to S_I. S_I's probability of accepting the equilibrium offer (which is the higher offer in this case) remains the same. If S_I rejects the higher offer (which in this case is the offer of $\delta^t H$ from B_1) and next period both the buyers make offers to S_M, then two periods from now, the remaining buyer offers $\delta^{t-2}H$ to S_I.

If B_2 deviates and offers $p^o \in (\delta^t H, \delta^{t-1}H]$ to S_I, then S_I rejects this with a probability that takes the updated belief to d_{t-2}. If S_I rejects this offer then next period if B_1 offers to S_I, he offers $\delta^{t-2}H$. If both B_1 and B_2 make offers to S_M then two periods from now the remaining buyer randomises between offering $\delta^{t-2}H$ and $\delta^{t-3}H$ to S_I (conditional on S_I being present). Randomisations are done in a manner to ensure that the expected continuation payoff to S_I from rejection is p^o. It is easy to check that for high values of δ, this can always be done. Lastly, if B_2 deviates and offers to S_I and B_1 offers to S_M (according to his equilibrium strategy), then the off-path specifications are the same as in the 2-player game with incomplete information.

We will now show that B_2 has no incentive to deviate. Suppose he makes an unacceptable offer to S_M. His expected discounted payoff from deviation is given by,

$$\mathcal{D} = q(\pi)[\delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\}] + (1 - q(\pi))\delta v_B(\pi) \tag{9}$$

From (4) we know that,

$$p_l^{'}(\pi) < M + \delta(1 - a(\pi))[\bar{p}(d_{t-1}) - M]$$

as $E_{d_{t-1}} < \bar{p}(d_{t-1})$. Hence we have,

$$p_l^{'}(\pi) < M + \delta(1 - a(\pi))[(v - M) - (v - \bar{p}(d_{t-1}))]$$

Rearranging the terms above we get,

$$(v - p_l^{'}(\pi)) > \delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\} + (1 - \delta)(v - M) \tag{10}$$

By comparing (9) and (10) we have,

$$q(\pi)(v - p_l^{'}(\pi)) + (1 - q(\pi))\delta v_B(\pi) > \mathcal{D}$$

The L.H.S of the above relation is B_2's equilibrium payoff, as he puts a mass point at $p_l^{'}(\pi)$. Hence he has no incentive to make an unacceptable offer to S_M.

Next, suppose B_2 deviates and makes an offer of p^o to S_I such that $p^o \in (\delta^t H, \delta^{t-1} H]$. B_2's payoff from deviation is:

$$\Gamma_H = q(\pi)[(v-p^o)a^{'}(\pi)+(1-a^{'}(\pi))\delta v_B(d_{t-2})]+(1-q(\pi))[(v-p^o)a(\pi)+(1-a(\pi))\delta v_B(d_{t-1})]$$

where $a^{'}(\pi)$ is the probability with which B_2's offer is accepted by S_I in the event when both B_1 and B_2 make offers to S_I and B_2's offer is in the range $(\delta^t H, \delta^{t-1} H]$. From our above specification it is clear that $a^{'}(\pi) > a(\pi)$, where $a(\pi)$ is the acceptance probability of an equilibrium offer to S_I. This is also very intuitive. In the contingency when B_1 makes an equilibrium offer to S_M and B_2's out of the equilibrium offer to S_I is in the range $(\delta^t H, \delta^{t-1} H]$, the acceptance probability is equal to $a(\pi)$, the equilibrium acceptance probability. In this case if the L-type S_I rejects an offer then next period he will get an offer with probability 1. However if both B_1 and B_2 make offers to S_I and B_2's offer is in the range $(\delta^t H, \delta^{t-1} H]$ then the L-type S_I accepts this offer with a higher probability. This is because, on rejection, there is a positive probability that S_I might not get an offer in the next period. This explains why $a^{'}(\pi) > a(\pi)$.

Since $p^o > p_l^{'}(\pi)$[21] and $\bar{p}(d_{t-2}) > p_l^{'}(\pi)$[22], we have

$$v - p_l^{'}(\pi) > (v - p^o)a^{'}(\pi) + (1 - a^{'}(\pi))\delta v_B(d_{t-2}) \tag{11}$$

Also, since $p^o > \delta^t H$, we have

$$(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1}) < v_B(\pi)$$

The expression $[(v-p^o)a(\pi)+(1-a(\pi))\delta v_B(d_{t-1})-\delta v_B(\pi)]$ is strictly negative for $\delta = 1$. From continuity, we can say that for sufficiently high values of δ, $(v-p^o)a(\pi)+(1-a(\pi))\delta v_B(d_{t-1}) < \delta v_B(\pi)$. This implies that,

$$(v - p_l^{'}(\pi))q(\pi) + (1 - q(\pi))\delta v_B(\pi) > \Gamma_H$$

[21] For sufficiently high values of δ this will always be the case.

[22] Since $\bar{p}(d_{t-2}) > \bar{p}(\pi) > p_l^{'}(\pi)$.

The L.H.S of the above inequality is the equilibrium payoff of B_2. Similarly if B_2 deviates and make an offer to S_I such that his offer p^0 is in the range $[\delta^{t+1}H, \delta^t H)$, the payoff from deviation is

$$\begin{aligned}\Gamma_L &= q(\pi)[\delta\{a(\pi)(v-M)+(1-a(\pi))v_B(d_{t-1})\}] \\ &\quad +(1-q(\pi))[(v-p^0)a^{''}(\pi)+(1-a^{''}(\pi))\delta v_B(d_t)]\end{aligned}$$

¿From the 2-player game we know that $[(v-p^0)a^{''}(\pi)+(1-a^{''}(\pi))\delta v_B(d_t)] < v_B(\pi)$. Also from the previous analysis we can posit that $(v-p_l^{'}(\pi)) > \delta\{a(\pi)(v-M)+(1-a(\pi))v_B(d_{t-1})\}$. Thus for sufficiently high values of δ, $(v-p_l^{'}(\pi))q(\pi)+(1-q(\pi))\delta v_B(\pi) > \Gamma_L$.

Hence B_2 has no incentive to deviate and make an offer to S_I.

C Off-path behavior with private offers

The off-path behavior described in the preceding appendix is not applicable to the case of private offers. This is because it requires the offers made by both the buyers to be publicly observable. The off-path behavior of the players in the case of private offers is described as follows.

Specifically we need to describe the behavior of the players in the following three contingencies.

(i) B_2 makes an unacceptable offer to S_M.

(ii) B_2 makes an offer of p^o to S_I such that $p^o < \delta^t H$.

(iii) B_2 makes an offer of p^o to S_I such that $p^o > \delta^t H$.

We denote the above three contingencies by E_1, E_2 and E_3 respectively. We now construct a particular belief system that sustains the equilibrium described in the text.

Suppose B_1 attaches probabilities λ,λ^2 and λ^3 ($0 < \lambda < 1$) to E_1, E_2 and E_3 respectively. Thus he thinks that B_2 is going to stick to his equilibrium behavior with probability $[1-(\lambda+\lambda^2+\lambda^3)]$.

If E_1 or E_2 occurs and B_1 makes an equilibrium offer to S_I, then S_I's probability of accepting the equilibrium offer remains the same and two periods from now (conditional on the fact that the game continues until then), if B_2 is the remaining buyer he offers $\delta^{t-2}H$ to S_I. If E_3 occurs and all players are observed to be present, then next period B_2 offers $\bar{p}(d_{t-1})$ to S_M. In any off-path contingency, if B_1 is the last buyer remaining (two periods from now) then he offers $\delta^{t-2}H$ to S_I.

The L-type S_I accepts an offer higher than $\delta^t H$ with probability 1 if she gets two offers. If she gets only one offer then the probability of her acceptance of out-of-equilibrium offers is the same as in the two-player game with incomplete information.

We will now argue that the off-path behavior constitutes a sequentially optimal response by the players to the limiting beliefs as $\lambda \to 0$.

Suppose B_1 makes an equilibrium offer to S_I and it gets rejected. Although offers are private, each player can observe the number of players remaining. Thus, next period, if B_1 finds that all four players are present he infers that this is due to an out-of-equilibrium play by B_2. Using Bayes' rule he attaches the following probabilities to E_1, E_2 and E_3 respectively.

$$\frac{1}{1+\lambda+\lambda^2} \text{ to } E_1$$
$$\frac{\lambda}{1+\lambda+\lambda^2} \text{ to } E_2$$
$$\frac{\lambda^2}{1+\lambda+\lambda^2} \text{ to } E_3$$

As $\lambda \to 0$, the probability attached to E_1 goes to 1. Thus B_1 believes that his equilibrium offer of $\delta^t H$ to S_I was rejected and the updated belief is d_{t-1}. In the case of E_1 or E_2 the beliefs of B_1 and B_2 coincide. However, in the case of E_3 they differ. Suppose E_3 occurs and B_1's equilibrium offer to S_I gets rejected. Then next period all four players will be present and given L-type S_I's behavior, the belief of B_2 will be $\pi = 0$ and that of B_1 will be $\pi = d_{t-1}$. In that contingency it is an optimal response of B_2 to offer $\bar{p}(d_{t-1})$ to S_M since he knows that B_1 is playing his equilibrium strategy with the belief d_{t-1}.

Next we will argue that the L-type S_I finds it optimal to accept an offer higher than $\delta^t H$ with probability 1, if she gets two offers. This is because in the event when she gets two offers she knows that rejection will lead the buyer B_1 to play according to the belief d_{t-1} and, two periods from now, the remaining buyer will offer $\delta^{t-2}H$ to S_I. Thus her continuation payoff from rejection is

$$\delta\{\delta^{t-1}Hq(d_{t-1}) + \delta(1-q(d_{t-1}))\delta^{t-2}H\} = \delta\{\delta^{t-1}H\} = \delta^t H$$

Hence she finds it optimal to accept an offer higher than $\delta^t H$ with probability 1.

We need to check that B_2 has no incentive to deviate and make an offer of p^o to S_I such that $p^o > \delta^t H$.

Suppose B_2 deviates and makes an offer of p^o to S_I such that $p^o > \delta^t H$. With probability $q(\pi)$, S_I will get two offers and $B_2's$ will be accepted with probability π. With probability

$(1-q(\pi))$, S_I will get only one offer. B_2 then gets a payoff of

$$(v-p^o)q(\pi)\pi+(1-q(\pi))[(v-p^o)a(\pi)+(1-a(\pi))\delta v_B(d_{t-1})]$$

As shown in the previous appendix, for high values of δ we have $(v-p^o)a(\pi)+(1-a(\pi))\delta v_B(d_{t-1})<\delta v_B(\pi)$. Also for high values of δ, $p^o>p_l^{'}(\pi)$. Thus[23],

$$\begin{aligned}v_B(\pi)&=(v-p_l^{'}(\pi))q(\pi)+(1-q(\pi))\delta v_B(\pi)\\&>(v-p^o)q(\pi)\pi+(1-q(\pi))[(v-p^o)a(\pi)+(1-a(\pi))\delta v_B(d_{t-1})]\end{aligned}$$

Hence B_2 has no incentive to deviate and make an offer of p^o to S_I.

Lastly, to show that B_2 has no incentive to deviate and make an unacceptable offer to S_M or offer p^0 to S_I such that $p^0<\delta^t H$ we refer to the analysis in the previous appendix.

D Off-path behavior for the 2-player game where the informed seller's valuation is drawn from a continuous distribution

Suppose the buyer makes an offer of p^0 such that $p^0>p^t$. We will show that for any $p^0\in(p^t,p^{t-1})$, the buyer will have no incentive to offer p^0. By definition, we have,

$$p^{t-1}-s^{t-2}=\delta(p^{t-2}-s^{t-2})\Rightarrow p^{t-1}-s^{t-1}>\delta(p^{t-2}-s^{t-1})$$

since $s^{t-1}<s^{t-2}$. Also,

$$p^t-s^{t-1}=\delta(p^{t-1}-s^{t-1})\Rightarrow p^t-s^{t-1}<\delta(p^{t-2}-s^{t-1})$$

since $p^{t-2}>p^{t-1}$. This implies that there exists a $\gamma\in(0,1)$ such that

$$\gamma p^{t-1}+(1-\gamma)p^t-s^{t-1}=\delta(p^{t-2}-s^{t-1})$$

Any $p^0\in(p^t,p^{t-1})$ can be written as $p^0=\eta p^{t-1}+(1-\eta)p^t$, where $\eta\in(0,1)$.

If $\eta<\gamma$ then rejection takes the posterior to s^{t-1}. The buyer following a rejection randomises between p^{t-1} and p^{t-2} such that the seller with valuation s^{t-1} is indifferent between accepting the offer of p^0 or rejecting it. Since $\eta<\gamma$, such a randomisation is always

[23] This is because B_2 puts a mass point at $p_l^{'}(\pi)$

possible. Also for the buyer, he is offering a higher price and it is getting accepted with the equilibrium probability.

If $\eta > \gamma$ then rejection takes the posterior to $s^{'} \in (s^{t-1}, s^{t-2})$ and the buyer next period offers p^{t-2}. Here $s^{'}$ is such that the seller with such a valuation is indifferent between accepting the offer of p^0 or rejecting it. Since $\eta > \gamma$,

$$p^0 - s^{t-1} > \delta(p^{t-2} - s^{t-1})$$

Also from definition, one can show that

$$p^0 - s^{t-2} < \delta(p^{t-2} - s^{t-1})$$

This shows that such a $s^{'}$exists.

Now, suppose the buyer offers some price p^0 such that $p^0 < p^t$. We will show that for any $p^0 \in (p^{t+1}, p^t)$, the buyer will have no incentive to deviate. For any $p^0 \in (p^{t+1}, p^t)$, there exists a $\alpha \in (0, 1)$ such that $p^0 = \alpha p^t + (1 - \alpha)p^{t+1}$. By definition, we have

$$p^{t+1} - s^t = \delta[p^t - s^t]$$

$$p^t - s^t > \delta[p^t - s^t]$$

Hence

$$p^0 - s^t > \delta[p^t - s^t]$$

Again by definition,

$$p^{t+1} - s^t = \delta[p^t - s^t] < \delta[p^{t-1} - s^t]$$

$$p^t - s^{t-1} = \delta[p^{t-1} - s^{t-1}] \Rightarrow p^t - s^t > \delta[p^{t-1} - s^t]$$

Hence there exists a $\gamma \in (0, 1)$ such that

$$\gamma p^t + (1 - \gamma)p^{t+1} - s^t = \delta[p^{t-1} - s^t]$$

Thus if $\alpha < \gamma$, then rejection takes the posterior to s^t. Next period the buyer randomises between offering p^t and p^{t-1}.

If $\alpha > \gamma$, then rejection takes the posterior to some $s^{'} \in (s^t, s^{t-1})$ such that a seller with valuation $s^{'}$ is indifferent between accepting the offer of p^0 or to reject it. As before it can be shown that such a $s^{'}$ exists.

E Out-of-equilibrium behavior for the 4-player game where the informed seller's valuation is drawn from a continuous support (public offers)

We only describe the following two off-path deviations. Others are analogous to the ones with the case where the informed seller's valuation is drawn from a distribution with two-point support.

First, suppose B_2 makes an unacceptable offer to S_M (i.e less than $p_l^{'}(s)$) and B_1 makes an equilibrium offer to S_I. Then rejection of the equilibrium offer by S_I still takes the posterior to s^{t-1}. However, next period, if B_1 offers to S_I, then he randomises between offering p^{t-1} and p^{t-2}. If next period, both the buyers offer to S_M, then two periods from now, the remaining buyer randomises between offering p^{t-1} and p^{t-2} to S_I. Note that when the posterior is s^{t-1}, the buyer is indifferent between offering p^{t-1} and p^{t-2}.

The payoff to the seller with valuation s^{t-1} from accepting an equilibrium offer now is $(p^t - s^{t-1})$. Hence randomisations by the buyers in the subsequent periods should ensure that the continuation payoff to the seller with valuation s^{t-1} from rejecting the equilibrium offer is also $(p^t - s^{t-1})$. We will now show that for high values of δ, such a randomisation is always possible.

Γ_l^c (the minimum continuation payoff to the seller with valuation s^{t-1}; i.e an offer of p^{t-1} in the next period and two periods from now.) is given as,

$$\Gamma_l^c = \delta[q(s^{t-1})(p^{t-1} - s^{t-1}) + (1 - q(s^{t-1}))\delta(p^{t-1} - s^{t-1})]$$

$$= \delta[p^{t-1} - s^{t-1}][q(.) + (1 - q(.))\delta] = (p^t - s^{t-1})[q(.) + (1 - q(.))\delta] < (p^t - s^{t-1})$$

(since by definition, $p^t - s^{t-1}) = \delta[p^{t-1} - s^{t-1}]$. This is true for all $\delta < 1$)

Γ_h^c (the maximum continuation payoff to the seller with valuation s^{t-1}; i.e an offer of p^{t-2} in the next period and two periods from now) is given as,

$$\Gamma_h^c = \delta[q(s^{t-1})(p^{t-2} - s^{t-1}) + (1 - q(s^{t-1}))\delta(p^{t-2} - s^{t-1})]$$

$$= \delta[(p^{t-2} - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)]$$

$$> \delta[(p^{t-1} - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)]$$

(since $p^{t-2} > p^{t-1}$)

$$= (p^{t} - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)$$

For $\delta = 1$ we have $\Gamma_h^c \geq (p^t - s^{t-1})$ (since $q(.) \to 0$, as $\delta \to 1$). This is because the inequality is strictly maintained when $\delta < 1$, and is not reversed when $\delta \to 1$ (as $p^{t-2} > p^{t-1}$ by definition) . Then by continuity we can say that for high values of δ, we will have $\Gamma_h^c > (p^t - s^{t-1})$. Also, we have $\Gamma_l^c < (p^t - s^{t-1})$. Hence on the equilibrium offer being rejected by the informed seller, offers to S_I can be made by randomising between p^{t-1} and p^{t-2} in a manner, such that the seller with valuation s^{t-1} is indifferent between accepting and rejecting the offer now. In the same way as done in the case of discrete valuations of the informed seller, one can show that the buyer B_2 has no incentive to deviate and make an unacceptable offer to S_M.

Next, suppose B_1 makes an equilibrium offer to S_I and B_2 deviates and makes an offer of p^0 to S_I, such that $p^0 < p^t$. Then the informed seller disregards the lower offer. Rejection takes the posterior to s^{t-1}. Thereafter buyers' behavior in making offers to S_I is exactly the same as described above.

Finally, suppose B_2 deviates and makes an offer of $p^0 > p^t$ to S_I and B_1 makes an equilibrium offer to S_I. Then rejection takes the posterior to s^{t-1}. We will show that for high values of δ, the buyer can always randomise between offering p^{t-1} and p^{t-2} in the next and subsequent periods (if there is no offer to S_I in the next period), such that the seller with valuation s^{t-1} is indifferent between accepting and rejecting the offer.

Any offer $p^0 \in (p^t, p^{t-1})$ is a convex combination of p^t and p^{t-1}. It is already shown above that the minimum continuation payoff to S_I with valuation s^{t-1}, $\Gamma_l^c < p^t - s^{t-1}$. Also,

$$\Gamma_h^c = \delta[q(s^{t-1})(p^{t-2} - s^{t-1}) + (1 - q(s^{t-1}))\delta(p^{t-2} - s^{t-1})]$$

$$= \delta[(p^{t-2} - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)]$$

$$> \delta[(p^{t-1} - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)]$$

Since the inequality is strictly maintained for $\delta < 1$ and not reversed when $\delta \to 1$, we have

$$\Gamma_h^c \geq p^{t-1} - s^{t-1}$$

for $\delta = 1$. Then by continuity we can posit that for high values of δ, we will have $\Gamma_h^c >$

$p^{t-1} - s^{t-1}$.

Hence the suggested randomisation is possible.

F Out-of-equilibrium behavior for the 4-player game where the informed seller's valuation is drawn from a continuous support (private offers)

Specifically we need to describe the behavior of the players in the following three contingencies:

(i) B_2 makes an unacceptable offer to S_M.

(ii) B_2 makes an offer of p^o to S_I such that $p^o < p^t$.

(iii) B_2 makes an offer of p^o to S_I such that $p^o > p^t$.

We denote the above three contingencies by E_1, E_2 and E_3 respectively. We now construct a particular belief system that sustains the equilibrium described in the text.

Suppose B_1 attaches probabilitites λ, λ^2 and λ^3 $(0 < \lambda < 1)$ to E_1, E_2 and E_3 respectively. Thus he thinks that B_2 is going to stick to his equilibrium behavior with probability $(1 - (\lambda + \lambda^2 + \lambda^3))$.

If E_1 or E_2 occurs and B_1 makes an equilibrium offer to S_I, then S_I's probability of accepting the equilibrium offer remains the same. On observing that all four players are present, the common posterior of the buyers will be s^{t-1}. In the subsequent periods when offers are made to S_I, randomisations between p^{t-1} and p^{t-2} are done in a manner to ensure that the continuation payoff to the informed seller with valuation s^{t-1} is $(p^t - s^{t-1})$. If E_3 occurs and all players are observed to be present, then next period B_2 offers $\bar{p}(s^{t-1})$ to S_M.

If the informed seller gets two offers, she accepts an offer $p^0 > p^t$ with probability 1 as long as her valuation is less than s'. Here s' is such that

$$p^0 - s' = p^t - s^{t-1}$$

If she gets only one offer then the probability of her acceptance of out-of-equilibrium offers is the same as in the two-player game with incomplete information.

We will now argue that the off-path behavior constitutes a sequentially optimal response by the players to the limiting beliefs as $\lambda \to 0$.

Suppose B_1 makes an equilibrium offer to S_I and it gets rejected. Although offers are private, each player can observe the number of players remaining. Thus, next period, if B_1 finds that all four players are present, he infers that this is due to an out-of-equilibrium play by B_2. Using Bayes' rule he attaches the following probabilities to E_1, E_2 and E_3 respectively.

$$\frac{1}{1+\lambda+\lambda^2} \text{ to } E_1$$

$$\frac{\lambda}{1+\lambda+\lambda^2} \text{ to } E_2$$

$$\frac{\lambda^2}{1+\lambda+\lambda^2} \text{ to } E_3$$

As $\lambda \to 0$, the probability attached to E_1 goes to 1. Thus B_1 believes that his equilibrium offer of p^t to S_I was rejected and the updated belief is s^{t-1}. In the case of E_1 or E_2 the beliefs of B_1 and B_2 coincide. However, in the case of E_3, they differ. Suppose E_3 occurs and B_1's equilibrium offer to S_I gets rejected. Then next period all four players will be present and given S_I's behavior, the belief of B_2 will be $s^{'} > s^{t-1}$ such that

$$p^0 - s^{'} = p^t - s^{t-1}$$

where $p^0 > p^t$ is the out of equilibrium offer made by B_2 to S_I (This in turn implies that the behavior of the informed seller in the contingency E_3 is optimal).

This is because the belief of B_1 is s^{t-1} and B_2, from the subsequent period onwards, plays according to B_1's belief. In the subsequent periods while offers are being made to S_I, randomisations between p^{t-1} and p^{t-2} are done in a manner to ensure that the continuation payoff to S_I is $p^t - s^{t-1}$. As before it is easy to observe that B_2 finds it optimal to play according to B_1's belief, since B_2's belief $(s^{'})$ is greater than that of B_1 (s^{t-1}) .

In the same way as done in the case of discrete valuations of the informed seller, we can show that B_2 will not deviate.

G Proof of lemma (21)

Proof. We only need to show that there exists a $\delta_t^* \geq \bar{\delta}$ such that for all $\delta > \delta_t^*$ and for all $s \in (s^{t+1}, s^t]$,there exists a $p_l(s) \in (p_l^{'}(s), \bar{p}(s))$ with

$$p_l(\pi) = (1-\delta)M + \delta E_s(p)$$

From now on we will write s_t instead of $s_t(\delta)$. For each $\delta \in (\delta^{'}, 1)$ we can construct $d(\delta)$ and the equilibrium strategies as above (assuming existence). Construct the function $G(x)$

as

$$G(x) = x - [\delta E_s^x(p) + (1-\delta)M]$$

From [7] we know that the function $G(.)$ is monotonically increasing in x. Since $E_\pi^x(p) < \bar{p}(\pi)$,

$$\lim_{x \to \bar{p}(\pi)} G(x) > 0$$

Next, we have

$$G(p_l^{'}(\pi)) = p_l^{'}(s) - [\delta E_s^{p_l^{'}(s)}(p) + (1-\delta)M]$$

By definition $E_\pi^{p_l^{'}(s)}(p) > p_l^{'}(s)$. So for $\delta = 1$, $G(p_l^{'}(\pi))) < 0$. Since $G(.)$ is a continuous function, there exists a $\delta_t^* \geq \bar{\delta}$ such that for all $\delta > \delta_t^*$, $G(p_l^{'}(\pi))) < 0$. By invoking the *Intermediate Value Theorem* we can say that there is a unique $x^* \in (p_l^{'}(\pi), \bar{p}(\pi))$ such that $G(x^*) = 0$. This x^* is our required $p_l(\pi)$.

This concludes the proof. ■

H Informed seller's reservation value is continuously distributed in $(L, H]$

Suppose the informed seller's valuation is continuously distributed on $(L, H]$ according to some cdf $G(s)$. As before, we first consider the two player game with a buyer and a seller, where the seller is informed.

H.1 Two-player Game

There is one buyer, whose valuation is commonly known to be v.

There is one seller, whose valuation is private information to her. Her valuation is distributed according to a continuous distribution function $G(.)$, over the interval $(L, H]$ with $L < H < v$.

Let $g(.)$ be the density function which is assumed to be bounded:

$$0 < \underline{g} \leq g(s) \leq \bar{g}$$

Players discount the future using a common discount factor $\delta \in (0, 1)$.

We now state the equilibrium of the infinite horizon bargaining game where the buyer makes offers in each period. The seller either accepts or rejects an offer. Rejection takes the game to the next period, when the buyer again makes an offer.

The result re-stated below (for completeness) is from [15] .

One can show that at any instant, the buyer's posterior distribution about the seller's valuation can be characterised by a unique number s^e, which is the lowest possible valuation of the seller. With a slight abuse of terminology, we will call s^e the buyer's posterior.

The Equilibrium: Given a $\delta \in (0,1)$, we can obtain thresholds s^t's, such that $L < s^t < H$ and

$$s^t < s^{t-1} < .... < s^2 < s^1$$

If at a time point t, the posterior $s_t \in (s^{t+1}, s^t]$, then the buyer offers p^t. A seller with valuation less than s^{t-1} accepts the offer. Rejection takes the posterior to s^{t-1}.

The p^t's are such that the seller with a valuation s^{t-1} is indifferent between accepting the offer now or waiting until the next period. The off-path behavior of players is outlined in appendix (D).

It can be shown that as $\delta \to 1$, for all t, $p^t \to H$. Also the maximum number of periods for which the game would last is bounded above by N^*.

H.2 Four-player game

We now analyse the four-player game. There are two buyers, each with a valuation v. There are two sellers. One of them has a valuation which is commonly known to be M. The other seller's valuation is private information to her. It is continuously distributed in $(L, H]$, according to some cdf $G(.)$ as discussed above.

Analogous to the discrete types case, we first show that if there exists a stationary equilibrium in the four-player game, then the asymptotic outcome for all such equilibria is unique. That is, as $\delta \to 1$, price offers in all transactions go to H. First, we prove the following proposition that establishes the result, conditional on a particular kind of equilibrium being ruled out. Later, we prove that this particular kind of equilibrium never exists.

Proposition 3 *Consider the set of stationary equilibria (possibly empty) of the four-player game such that any equilibrium belonging to this set has the property that both buyers do not make offers only to the informed seller (S_I) on the equilibrium path. As the discount factor $\delta \to 1$, all price offers in any equilibrium belonging to this set converge to H.*

Proof. We prove this proposition in steps, through a series of lemmas. First, we show that for any equilibrium belonging to the set of equilibria considered, the following lemma holds.

Lemma 12 *For any s, it is never possible to have a stationary equilibrium in the set of equilibria considered such that both buyers offer only to S_M on the equilibrium path.*

Proof. Suppose it is the case that both buyers offer only to S_M. Since this is a stationary equilibrium, both buyers should have a distribution of offers to S_M with a common support $[\underline{p}(s), \bar{p}(s)]$. The payoff to each buyer should then be $(v - \bar{p}(s)) = v_4(s)$(say). Let $v_B(s)$ be the payoff obtained by a buyer when his offer to S_M gets rejected.

Consider any $p \in [\underline{p}(s), \bar{p}(s)]$ and one of the buyers (say B_1). If the distributions of the offers are given by F_i for buyer i, then we have

$$(v - p)F_2(p) + (1 - F_2(p))\delta v_B(s) = v - \bar{p}(s)$$

Since in equilibrium, the above needs to be true for any $p \in [\underline{p}(s), \bar{p}(s)]$, we must have $v - \bar{p}(s) > \delta v_B(s)$. The above equality gives us

$$F_2(p) = \frac{(v - \bar{p}(s)) - \delta v_B(s)}{(v - p) - \delta v_B(s)}$$

Since $v - \bar{p}(s) > \delta v_B(s)$, for $p \in [\underline{p}(s), \bar{p}(s))$, we have $v - p > v - \bar{p}(s) > \delta v_B(s)$. This would imply

$$F_2(\underline{p}(s)) > 0$$

Similarly, we can show that

$$F_1(\underline{p}(s)) > 0$$

In a stationary equilibrium, it is not possible for both the buyers to put mass points at the lower bound of the support. Hence, S_M cannot get two offers with probability 1. This concludes the proof of the lemma. ■

For any equilibrium belonging to the set of equilibria we are considering, we know that S_M must get at least one offer with positive probability. The above lemma implies that S_I also gets at least one offer with a positive probability. We will now argue that for any equilibrium in the set of equilibria considered, S_M always accepts an equilibrium offer immediately. This is irrespective of whether S_M gets one offer or two offers.

To show this formally, consider such an equilibrium. We first define the following. Given a s, let $p_i(s)$ be the minimum acceptable price to the seller S_M in the event she gets i $(i = 1, 2)$ offer(s) in the considered equilibrium. We have

$$p_1(s) - M = (1 - (\alpha(s))\delta[E_p(\tilde{s}) - M]$$

$E_p(\tilde{s})$ is the price corresponding to the expected equilibrium payoff to the seller S_M in the event she rejects the offer and the informed seller does not accept the offer. It is evident that when the seller S_M is getting one offer, the informed seller is also getting an offer. Here

$\alpha(s)$ is the probability with which the informed seller accepts the offer and $\tilde{s}$ is the updated posterior.

Similarly, we have

$$p_2(s) - M = \delta[E_p(s) - M]$$

where $E_p(s)$ is the price corresponding to the expected equilibrium payoff to S_M in the event she rejects both offers. Similar to the discrete types case, we can argue that $E_p(s) > M$. The following lemma has the consequence that S_M always accepts an equilibrium offer (or highest of the equilibrium offers) immediately.

Lemma 13 *For any $s < H$, if we restrict ourselves to the set of equilibria considered, then in any arbitrary equilibrium, it is never possible for a buyer to make an offer to S_M that is strictly less than* $\min\{p_1(s), p_2(s)\}$.

Proof. Suppose the conclusion of the lemma does not hold, so there is such an equilibrium. Let the payoff to the buyers from this candidate equilibrium of the four-player game be $v_4(s)$. As in the discrete types case, we argue that $v_4(s) < v - p_2(s)$. Let $v_B(s)$ be the payoff the buyer gets by making offers to S_I in a two-player game.

Consider the buyer who makes the lowest offer to S_M. We label this buyer as B_1 and the lowest offer as $\underline{p(s)}$,where $\underline{p(s)} < \min\{p_1(s), p_2(s)\}$. Let $q(s)$ be the probability with which the other buyer makes an offer to the seller S_I. Let $\gamma(s)$ be the probability with which the other buyer, conditional on making offers to the seller S_M, makes an offer which is less than $p_2(\pi)$. Finally, $\alpha(s)$ is the probability with which the informed seller accepts an offer if the other buyer makes an offer to her. Since B_1's offer of $\underline{p(s)}$ to S_M is always rejected, the payoff to B_1 from making such an offer is

$$\{q(s)\delta\{\alpha(s)(v - M) + (1 - \alpha(s))(v - E_p^b(\tilde{s}))\} + (1 - q(s))\delta\{\gamma(s)v_4(s) + (1 - \gamma(s))v_B(s)\}$$

where $E_p^b(\tilde{s})$ is such that $(v - E_p^b(\tilde{s}))$ is the expected equilibrium payoff to the buyer if the updated belief is $\tilde{s}$. We first argue that $(v - E_p^b(\tilde{s}))$ is less than or equal to $(v - E_p(\tilde{s}))$. This is because since $(E_p(\tilde{s}) - M)$ is the expected equilibrium payoff to the seller S_M when the belief is $\tilde{s}$, there is at least one price offer by the buyer that is greater than or equal to $E_p(\tilde{s})$. Hence, we have

$$\delta\{\alpha(s)(v - M) + (1 - \alpha(s))(v - E_p^b(\tilde{s}))\} \leq \delta\{\alpha(s)(v - M) + (1 - \alpha(s))(v - E_p(\tilde{s}))\}$$

$$\Rightarrow (v - p_1(s)) - \delta\{\alpha(s)(v - M) + (1 - \alpha(s))(v - E_p^b(\tilde{s}))\}$$

$$\geq (v - p_1(s)) - \delta\{\alpha(s)(v - M) + (1 - \alpha(s))(v - E_p(\tilde{s}))\}$$

Since $(v - p_1(s)) - \delta\{\alpha(s)(v - M) + (1 - \alpha(s))(v - E_p(\tilde{s}))\} = (1 - \delta)(v - M) > 0$, we have

$$(v - p_1(s)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha(s))(v - E^b_p(\tilde{s}))\} > 0$$

There are two possibilities. Either $p_1(s) < p_2(s)$ or $p_2(s) < p_1(s)$. If $p_2(s) > p_1(s)$, then the buyer can profitably deviate by making an offer of $p_1(s)$. The payoff from making such an offer is

$$q(s)(v - p_1(s)) + (1 - q(s))\{\gamma(s)\delta v_4(s) + (1 - \gamma(s))\delta v_B(s)\}$$

Since $(v - p_1(s)) - \delta\{\alpha(s)(v - M) + (1 - \alpha(s))(v - E^b_p(\tilde{s}))\} > 0$, we can infer that this constitutes a profitable deviation by the buyer.

Next, consider the case when $p_2(s) < p_1(s)$. In this situation, the buyer can profitably deviate by making an offer of $p_2(s)$. The payoff from making such an offer is

$$\{q(s)\delta\{\alpha(s)(v - M) + (1 - \alpha(s))(v - E^b_p(\tilde{s}))\} + (1 - q(s))\{\gamma(s)(v - p_2(s)) + (1 - \gamma(s))\delta v_B(s)\}$$

Since $v_4(s) < (v - p_2(s))$, this constitutes a profitable deviation by the buyer.

This concludes the proof of the lemma. ■

There are two immediate conclusions from the above lemma. First, if $p_2(s) < p_1(s)$, then it can be shown that if δ is high enough, then in equilibrium, no buyer should offer anything less than $p_1(s)$. To show this, suppose at least one of the buyers makes an offer which is less than $p_1(s)$ and consider the buyer who makes the lowest offer to S_M. Let $\gamma_1(s)$ be the probability with which the other buyer, conditional on making offers to S_M, makes an offer which is less than $p_1(s)$. The payoff to the buyer by making the lowest offer to S_M is

$$\{q(s)\delta\{\alpha(s)(v - M) + (1 - \alpha(s))(v - E^b_p(\tilde{s}))\} + (1 - q(s))\delta\{v_B(s)\}$$

However, if he makes an offer of $p_1(s)$ then the payoff is

$$\{q(s)(v - p_1(s)) + (1 - q(s))\{\gamma_1(s)(v - p_1(s)) + (1 - \gamma_1(s))\delta v_B(s)\}$$

We know that as $\delta \to 1$, $v_B(s) \to v - H$. Since $p_1(s) < H$, this implies that for high δ, $\gamma_1(s)(v - p_1(s)) + (1 - \gamma_1(s))\delta v_B(s) > \delta v_B(s)$. Hence, for high δ, this constitutes a profitable deviation by the buyer.

Secondly, if $p_1(s) < p_2(s)$, then only one buyer can make an offer with positive probability that is less than $p_2(s)$. This is because, any buyer who makes an offer to S_M in the range $(p_1(s), p_2(s))$ can get the offer accepted when the seller S_M gets only one offer. In that case the offer can still get accepted if it is lowered and that would not alter the outcomes following the rejection of the offer. Hence, the buyer can profitably deviate by making a lower offer. Thus, in equilibrium if a buyer has to offer anything less than $p_2(s)$ to the seller S_M, then it has to be equal to $p_1(s)$. However, in equilibrium both buyers cannot put a mass point at $p_1(s)$. This shows that only one buyer can make an offer to S_M which is strictly less than $p_2(s)$.

Hence, we have argued that all offers to S_M are always greater than or equal to $p_1(s)$ and in the event S_M gets two offers, both offers are never below $p_2(s)$. This shows that S_M always accepts an equilibrium offer immediately.

We will now show that for any equilibrium in the set of equilibria considered, the informed seller by rejecting equilibrium offers for a finite number of periods can take the posterior to H. This is shown in the following lemma.

Lemma 14 *Suppose we restrict ourselves to the set of equilibria considered. Given a s and δ, there exists a $T_s(\delta) > 0$ such that, conditional on getting offers, the informed seller can get an offer of H in $T_s(\delta)$ periods from now by rejecting all offers she gets in between. $T_s(\delta)$ depends on the sequence of equilibrium offers and corresponding strategies of the responders in the candidate equilibrium. $T_s(\delta)$ is uniformly bounded above as $\delta \to 1$.*

Proof.

In any equilibrium, a positive mass of types of the informed seller should always accept. If a particular type s accepts, then all types $s^{'} < s$ should also accept. Thus, rejection always leads to an upward revision of the posterior. This proves the first part of the lemma.

We will show that given a posterior s, in any equilibrium, the mass of informed seller accepting the lowest offer is always bounded below as $\delta \to 1$.

As argued in the discrete types case, we only need to consider the case where two buyers offer to S_I with positive probability. For a given s, let p_l be the minimum offer which gets accepted by a positive mass of types of S_I. Like the discrete types case, we know that there exists a possible outcome such that S_I gets the offer of p_l only. Since, S_M always accepts an equilibrium offer immediately, S_I in such a situation knows that rejecting the offer will lead to a two- player game. By invoking the finiteness property of the two-player game we can infer that as $\delta \to 1$, the number of rejections required for the informed seller to get an offer of H converges to some $\tilde{T}$ ($0 < \tilde{T} < \infty$). Further, posteriors in each of the time periods $t = 1, .., \tilde{T} - 1$ also reaches a limit s_t. Let p_t be the price offered in time period $t > 0$, following a rejection from the informed seller in $t - 1$. Note that $p_0 = p_l$ and $p_{\tilde{T}} = H$.

Suppose there exists an equilibrium of the four-player game such that p_l is the initial price offer and the number of rejections required for the informed seller to get H is not bounded above as $\delta \to 1$. This implies that we can find a $\delta' < 1$ such that for all $\delta > \delta'$, we have $s_t^u < s_t$ for all $t = 1, 2, ..., \tilde{T}$. s_t^u is the updated posterior following a rejection in $t-1$. Let p_t^u be the offer (or highest of the offers) to the informed seller at time point $t \geq 1$.

We will now show that for all $t \geq 1$, $p_t^u > p_t$.

Consider $t = 1$. We know that

$$p_l - s_1 = \delta(p_1 - s_1) \Rightarrow \delta p_1 = p_l - (1-\delta)s_1$$

Similarly,

$$p_l - s_1^u = \delta(p_1^u - s_1^u) \Rightarrow \delta p_1^u = p_l - (1-\delta)s_1^u$$

Thus, we have $p_1^u - p_1 = (1-\delta)(s_1 - s_1^u) > 0 \Rightarrow p_1^u > p_1$.

Consider $t > 1$. We will show that if $p_{t-1}^u > p_{t-1}$, then $p_t^u > p_t$.

$$p_{t-1} - s_t = \delta(p_t - s_t) \Rightarrow \delta p_t = p_{t-1} - (1-\delta)s_t$$

Similarly,

$$p_{t-1} - s_t^u = \delta(p_t^u - s_t^u) \Rightarrow \delta p_t^u = p_{t-1}^u - (1-\delta)s_t^u$$

Thus, we have $p_t^u - p_t = p_{t-1}^u - p_{t-1} + (1-\delta)(s_t - s_t^u) > 0$. Hence for all $t \geq 1$, $p_t^u > p_t$.

This implies that $p_{\tilde{T}} \geq H$. Hence, the price offer reaches H after finite number of rejections. This is contradictory to our conjectured hypothesis. Thus, in any equilibrium, the informed seller can get an offer of H after rejecting for finite number of times. This concludes the proof of the lemma. ■

We will now show that in a stationary equilibrium, both buyers cannot make offers to both sellers with positive probability.

Lemma 15 *In any equilibrium belonging to the set of equilibria considered, if players are patient enough then both buyers cannot make offers to both sellers with positive probability.*

Proof. In equilibrium, any offer made to the informed seller should get accepted by a positive mass of types of S_I. Suppose there exists a stationary equilibrium of the four-player game where both buyers offer to both sellers with a positive probability. Hence, in equilibrium, if the informed seller gets offer(s), then she either gets two offers or one offer. Since S_M always accepts an offer in equilibrium immediately, S_I knows that on rejecting an offer(s) she will get another offer in at most two periods from now. Hence, from lemma (14) we infer that if the informed seller gets one offer, then the type s-S_I who accepts the offer now can expect to get an offer of H in at most $T_1(s) > 0$ time periods from now, by rejecting all offers she gets in between. Similarly, if the informed seller gets two offers then the type s-S_I who accepts an offer now by rejecting both offers can expect to get an offer of H in at most $T_2(s) > 0$ time periods from now by rejecting all offers she gets in between. As we have argued in lemma (14), both $T_1(s)$ and $T_2(s)$ are bounded above as $\delta \to 1$. Thus, any offer p to the informed seller in equilibrium should satisfy

$$p \geq \delta^{T_1(s)} H + (1 - \delta^{T_1(s)}) s \equiv p_1(\delta)$$

and

$$p \geq \delta^{T_2(s)} H + (1 - \delta^{T_2(s)}) s \equiv p_2(\delta)$$

It is clear from the above that as $\delta \to 1$, both $p_1(\delta) \to H$ and $p_2(\delta) \to H$. Hence, if there is a non-degenerate support of offers to S_I in equilibrium, then the support should collapse as $\delta \to 1$.

We will now argue that for δ high enough but $\delta < 1$, the support in equilibrium cannot have two or more points.

Suppose it is possible that the support of offers to S_I has two or more points. This implies that the upper bound and the lower bound of the support are different from each other. Let $\underline{p}(s)$ and $\bar{p}(s)$ be the lower and upper bound of the support respectively.

Consider a buyer who is making an offer to S_I. This buyer must be indifferent between making an offer of $\underline{p}(s)$ and $\bar{p}(s)$. Let $q(s)$ be the probability with which the other buyer makes an offer to S_I. Since in equilibrium S_M always accepts an offer immediately, the payoff from making an offer of $\underline{p}(s)$ to S_I is

$$\begin{aligned} \Pi_{\underline{p}(s)} &= (1 - q(\pi))[\alpha_{\underline{p}(s)}(v - \underline{p}(s)) + (1 - \alpha_{\underline{p}(s)})\delta v_B(s^{'})] \\ &\quad + q(s) E_p\{[\beta_s^p \delta(v - M) + (1 - \beta_s^p)\delta v_4(s_p^{''})]\} \end{aligned}$$

$\alpha_{\underline{p}(s)}$ is the probability of acceptance of the offer $\underline{p}(s)$ when S_I gets the offer of $\underline{p}(s)$ only. β_s^p is the acceptance probability of the offer p when S_I gets two offers. $v_B(.)$ and $v_4(.)$ are the buyer's payoffs from the two-player incomplete information game and the four player incomplete information game respectively. For the second term of the right-hand side, we have taken an expectation because when two offers are made, this buyer's offer of $\underline{p}(s)$ to S_I never gets accepted and the payoff then depends on the offer made by the other buyer. When S_I gets only one offer and rejects an offer of $\underline{p}(s)$, then the updated posterior is $s^{'}$; $s_p^{''}$ denotes the updated posterior when S_I rejects an offer of $p \in (\underline{p}(s), \bar{p}(s)]$ and she gets two offers.

Similarly, the payoff from offering $\bar{p}(s)$ is

$$\Pi_{\bar{p}(s)} = (1-q(s))[\alpha_{\bar{p}(s)}(v-\bar{p}(s)) + (1-\alpha_{\bar{p}(s)})\delta v_B(s^{'''})] + q(s)[\beta_{2s}(v-\bar{p}(s)) + (1-\beta_{2s})\delta v_4(s^4)]$$

Here $s^{'''}$ is the updated posterior when S_I gets one offer and rejects an offer of $\bar{p}(s)$. When S_I gets two offers and rejects an offer of $\bar{p}(s)$, the updated posterior is denoted by s^4. Note that if at all S_I accepts an offer, she always accepts the offer of $\bar{p}(s)$, if made. The quantity $\alpha_{\bar{p}(s)}$ is the probability with which the offer of $\bar{p}(s)$ is accepted by S_I when she gets one offer. When S_I gets two offers, then the offer of $\bar{p}(s)$ gets accepted with probability β_{2s}.

As argued above, $\bar{p}(s) \to H$ and $\underline{p}(s) \to H$ as $\delta \to 1$. This implies that $v_4(s) \to (v-H)$ as $\delta \to 1$. From the result of the two player one-sided asymmetric information game, we know that $v_B(s) \to H$ as $\delta \to 1$. Since $v-M > v-H$, we have $\Pi_{\underline{p}(s)} > \Pi_{\bar{p}(s)}$ as $\delta \to 1$. From lemma (3) we can infer that both β_s^p's and β_{2s} are positive. Hence, there exists a threshold for δ such that if δ crosses that threshold, $\Pi_{\underline{p}(s)} > \Pi_{\bar{p}(s)}$. This is not possible in equilibrium. Thus, for high δ, the support of offers can have only one point. The same arguments hold for the other buyer as well. Hence, each buyer while offering to S_I has a one-point support. Next, we establish that both buyers should make the same offer. If they make different offers, then as explained before, for δ high enough the buyer making the higher offer can profitably deviate by making the lower offer. However, in equilibrium it is not possible to have both buyers making the same offer to S_I[24]

Hence, when agents are patient enough, in equilibrium both buyers cannot offer to both sellers with a positive probability. This concludes the proof of the lemma. ■

In the following lemma we show that in any stationary equilibrium of the four player game, as players get patient enough, S_M always gets offers from two buyers with a positive probability.

Lemma 16 *In any stationary equilibrium belonging to the set of equilibria considered, there exists a threshold of δ such that if δ exceeds that threshold, both buyers make offers to S_M with positive probability.*

Proof. Suppose there exists a stationary equilibrium where S_M gets offers from only one buyer , say B_1. First, we argue that in such a stationary equilibrium, if the buyer offering to S_M offers something greater than or equal to M, then S_M accepts it immediately. To explain this, let $p_m \geq M$ be the offer made by the buyer who makes offers to S_M. Then, S_M

[24]These arguments would also work even if the supports were not taken to be symmetric. In that case, let $\underline{p}(s)$ be the minimum of the lower bounds and $\bar{p}(s)$ be the maximum of the upper bounds. If these are associated with the same buyer, then same arguments hold. If not, then the buyer with the higher upper bound can proftibaly deviate by shifting its mass to $\underline{p}(s)$.

on rejecting this offer either gets back a two- player game or a four-player game. In either case, she cannot expect to get anything more than p_m. Hence, she immediately accepts it. This implies that if there is a stationary equilibrium where S_M gets offers from only one buyer then that buyer should always offer M to S_M and S_M immediately accepts it.

There can, therefore, be two possibilities. Either S_I gets an offer only from B_2 or from both B_1 and B_2 with positive probability. Consider the first case. Since S_M will accept the offer immediately and some s-type S_I should always accept, B_2, must be making an offer greater than or equal to p^e, such that

$$p^e = (1-\delta)s + \delta(H-\epsilon)$$

where $\epsilon > 0$ and $\epsilon \to 0$ as $\delta \to 1$. This is because in equilibrium, if S_I rejects an offer then next period she faces a two-player game. This game has a unique equilibrium and the price offers in that equilibrium goes to H as $\delta \to 1$.

From this we can infer that there exists a threshold of δ such that if δ exceeds that threshold then $p^e > M$.

Hence, B_2 can profitably deviate, contradicting the hypothesis of equilibrium.

In the latter case,we know that B_1 offers to both S_I and S_M with positive probability and B_2 makes offers only to S_I. To get the offer accepted by a positive mass of sellers, for high values of δ that offer should be close to H and thus the payoff to B_1 from making offers to S_I should be close to $(v-H)$. On the other hand, the payoff to B_1 from making offers to S_M is $(v-M)$. However, in equilibrium, the buyer has to be indifferent between making offers to S_I and S_M. Hence, it is not possible to have a stationary equilibrium where S_M gets offers from only one buyer. This concludes the proof. ■

From the characteristics of the restricted set of equilibria being considered, we know that S_M always gets an offer with a positive probability. The above lemma then allows us to infer that, in any stationary equilibrium of the four player game, both buyers should offer to S_M with positive probability. From our arguments and hypothesis, we know that both buyers cannot make offers to only one seller (S_I or S_M) and both buyers cannot randomise between making offers to both sellers. Hence, we can infer that one of the buyers has to make offers to S_M only and the other buyer should randomise between making offers to S_I and S_M.

The following lemma now shows that for any $s \in (L, H]$, any equilibrium in this restricted set possesses the characteristic that the price offers to all sellers approach H as $\delta \to 1$.

Lemma 17 *For a given s, in any hypothesised equilibrium, price offers to all sellers go to H as $\delta \to 1$.*

Proof. Let $\bar{p}(s)$ be the upper bound of the support of offers to S_M.

S_M always accepts an equilibrium offer immediately. Let $s^{'}$ be the updated posterior following a rejection by the informed seller. Hence, if the $s^{'}$-type S_I rejects an equilibrium offer, she gets back a two-player game with one-sided asymmetric information. Thus, the buyer offering to S_I in a period must offer at least p^e such that

$$p^e - s^{'} = \delta(H - \epsilon - s^{'}) \Rightarrow p^e = (1-\delta)s^{'} + \delta(H-\epsilon)$$

where $\epsilon > 0$ and $\epsilon \to 0$ as $\delta \to 1$.

Consider B_1, who is randomising between making offers to S_I and S_M. When offering to S_I, B_1 must offer p^e and it must be the case that

$$(v - p^e)\alpha(s) + (1 - \alpha(s))\delta\{v - (H - \epsilon)\} = v - \bar{p}(s)$$

where $\alpha(s)$ is the probability with which the offer is accepted by the informed seller. This follows from the fact that B_1 must be indifferent between offering to S_I and S_M. The L.H.S of the above equality is the payoff to B_1 from offering to S_I and the R.H.S is the payoff to him from offering to S_M. Since in any hypothesized equilibrium, S_M always gets an offer in period 1 and S_M accepts an equilibrium offer immediately, S_I, by rejecting an equilibrium offer, always gets back a two-player game with one-sided asymmetric information. Hence, the payoff to the buyer from offering to S_I is the same as in the two-player game with one-sided asymmetric information. This implies that

$$(v - p^e)\alpha(s) + (1 - \alpha(s))\delta\{v - (H - \epsilon)\} = v_B(s)$$

Thus, we can conclude that $v_B(s) = v - \bar{p}(s)$.

We will now show that the upper bound of the support of offers to S_M is strictly greater than p^e. We have

$$(v - p^e) - \delta\{v - (H - \epsilon)\} = v(1-\delta) + \delta(H-\epsilon) - \delta(H-\epsilon) - (1-\delta)s^{'}$$

$$= (1-\delta)(v - s^{'}) > 0$$

for $\delta < 1$. This implies that

$$v - p^e > \delta\{v - (H - \epsilon)\}$$

Since $(v - \bar{p}(s))$ is a convex combination of $v - p^e$ and $\delta\{v - (H - \epsilon)\}$, we have

$$v - p^e > v - \bar{p}(s^{'}) \Rightarrow \bar{p}(s) > p^e$$

Next, we will argue that as $\delta \to 1$, the support of offers to S_M from any buyer is bounded below by p^e. Consider a buyer who makes an offer of p^e to S_M in equilibrium. Then, if q^e is the probability with which this offer gets accepted, we have

$$(v - p^e)q^e + (1 - q^e)\delta v_B(s) = v - \bar{p}(s)$$

This follows since S_M always accepts an offer in equilibrium immediately, this buyer's offer to S_M gets rejected only when the other buyer also makes an offer to S_M.

This gives us,

$$q^e = \frac{(v - \bar{p}(s)) - \delta v_B(s)}{(v - p^e) - \delta v_B(s)} = \frac{(1-\delta)(v - \bar{p}(s))}{(v - p^e) - \delta(v - \bar{p}(s))}$$

$$\Rightarrow q^e = \frac{1}{\frac{v}{v-\bar{p}(s)} + \frac{\delta\bar{p}(s) - p^e}{(1-\delta)(v-\bar{p}(s))}}$$

and

$$q^e \to 0 \text{ as } \delta \to 1$$

This shows that in equilibrium, as $\delta \to 1$, any offer to S_M that is less than or equal to p^e always gets rejected. Since we have argued earlier that in equilibrium, no buyer should make an offer to S_M that she always rejects, we can infer that the support of offers to S_M from any buyer is bounded below by p^e as δ approaches 1. Hence, in any arbitrary stationary equilibrium of this kind, the price offers to all sellers are bounded below by p^e as δ approaches 1. However, as $\delta \to 1$, $p^e \to H$. Hence, as $\delta \to 1$, the support of offers to S_M from any buyer collapses and hence price offers to all sellers converge to H. ■

Thus, we have shown that for any stationary equilibrium in the set of equilibria considered, one of the buyers randomises between making offers to S_M and S_I and the other buyer makes offers to S_M only. Further, as $\delta \to 1$, price offers in all transactions in these stationary equilibria go to H. This concludes the proof of the proposition. ■

We will now argue that there does not exist any stationary equilibrium where both buyers offer only to S_I. This is done in the following lemma.

Lemma 18 *Let S be the set of all posteriors ($s < H$) such that for $s \in S$, it is possible to have a stationary equilibrium where both buyers offer only to S_I. The set S is empty*

Proof. We begin the proof by first showing that the set S^c is non-empty. Suppose not. Then, for all s, it is possible to have a stationary equilibrium where both buyers make offers only to S_I. In this case, price offers can never exceed M. This is because S_M does not get any offer in the presence of all four players. Thus, if any buyer unilaterally deviates and offers a price greater than or equal to M to S_M, S_M will accept it.

Let $\bar{p}$ be the largest price offer, for any s in such an equilibrium. (Clearly, such an offer exists.) This offer is accepted by any $s^{'}$ -type S_I ($s \leq s^{'} < H$) with probability 1, since the payoff from rejecting can be at most $\delta(\bar{p} - s^{'})$ and $\delta < 1$. But then, in the following period, the posterior is H (by Bayes' Theorem) and, therefore, as $\delta \to 1$, the payoff to S_I from such a continuation game is close to $H - s^{'}$ ($> M - s^{'}$). Since $\bar{p} \leq M$, the $s^{'}$-type S_I can unilaterally deviate to get a higher payoff and hence, this cannot be an equilibrium. This shows that S^c is non-empty.

Suppose now that S is non-empty. Consider any $s \in S$. No equilibrium can involve offers that are rejected by all $s < H$ types. Therefore, some $s < H$ types must accept an offer with positive probability. This implies (by Bayes' Theorem and $\delta < 1$) that the sequence of prices must be increasing. Also, by hypothesis, the price is bounded above by M. Let $\bar{p}^{'}$ be the largest price offer in such an equilibrium. As argued before, $\bar{p}^{'} \leq M$. There are two possibilities. Either the updated posterior conditional on $\bar{p}^{'}$ being rejected is in S or it is in S^c. In the former case, S_I should accept the offer with probability 1 and the updated posterior is H where the equilibrium price offer must be $H > M$, leading to the existence of a profitable deviation, for δ sufficiently high. For the latter case, if δ is high then from proposition (3) we know that for any stationary equilibrium all offers converge to H as $\delta \to 1$. Once again, this implies the existence of a profitable deviation for the S_I of type s which accepts the equilbrium offer. Hence, we cannot have S non-empty. This concludes the proof. ■

We now state our main result of the paper in the theorem below

Theorem 2 *In any arbitrary stationary equilibrium of the four-player game, as the discount factor goes to* 1*, price offers in all transactions converge to* H *for all values* $s \in [L, H)$*.*

Proof. The proof the theorem follows directly from proposition (3) and lemma (18). ■

H.2.1 Characterisation of a stationary equilibrium

We start this subsection by proving an analogue of the *competition lemma*. From the two-player game, we know that the number of periods for which the game with one-sided asymmetric information would last is bounded above by N^*.

Lemma 19 *For* $t \geq 1, ..., N^*$*, define* $\bar{p}_t$ *and* $p_t^{'}$ *as*

$$\bar{p}_t = v - [(v - p^t)\alpha + (1 - \alpha)\delta(v - \bar{p}_{t-1})]$$

$$p_t^{'} = M + \delta(1-\alpha)(\bar{p}_{t-1} - M)$$

where $\alpha \in (0,1)$ *and* $\bar{p}_0 = H$.

Then there exists $\delta^{'} \in (0,1)$ *such that for* $\delta > \delta^{'}$ *and for all* $t \in \{1, 2, 3, ..., N^*\}$ *we have*

$$\bar{p}_t > p_t^{'}$$

Proof.

$$\bar{p}_t - p_t^{'} = v - [(v - p^t)\alpha + (1-\alpha)\delta(v - \bar{p}_{t-1})]$$

$$-M - \delta(1-\alpha)(\bar{p}_{t-1} - M)$$

$$= (1-\delta)(v - M) + \alpha[p^t - \delta M - (1-\delta)v]$$

The first term is always positive. Let us consider the second term. The coefficient of α is positive for $\delta = 1$. This is because $p^t \to H$ as $\delta \to 1$. Hence, there exists a $\delta^{'}$ such that when $\delta > \delta^{'}$, the term is positive.

This concludes the proof. ■

For each $\delta \in (0,1)$ we can find a t such that $s \in (s^{t+1}, s^t]$. The sequence $\{s^{t+1}, s^t, ..., s^3, s^2\}$ is derived from and is identical with the same sequence in the two player game. Given these, we can evaluate $v_B(s)$ as

$$(v - p^t)\frac{[G(s^{t-1}) - G(s)]}{1 - G(s)} + \frac{1 - G(s^{t-1})}{1 - G(s)}\delta(v_B(s^{t-1}))$$

for $s \leq s^2$.

For $s > s^2$, $v_B(s) = v - H$.

Define $\bar{p}(s)$ as,

$$\bar{p}(s) = v - v_B(s)$$

As before, we first conjecture an equilibrium and derive it and then prove existence. We refer to the seller with known valuation as S_M and the one with private information as S_I.

The following proposition describes the equilibrium.

Proposition 4 *There exists a* $\delta^* \in (0,1)$ *such that if* $\delta > \delta^*$, *then for all* $s \in (L, H)$ *there exists a stationary perfect Bayes' equilibrium as follows:*

(i) One of the buyers (say B_1*) will make offers to both* S_I *and* S_M *with positive probability. The other buyer* B_2 *will make offers to* S_M *only.*

(ii) B_2 *while making offers to* S_M *will put a mass point at* $p_l^{'}(s)$ *and will have an absolutely continuous distribution of offers from* $p_l(s)$ *to* $\bar{p}(s)$ *where* $p_l^{'}(s)$ $(p_l(s))$ *is the minimum acceptable price to* S_M *when she gets one (two) offer(s). For a given* s, $\bar{p}(s)$ *is the upper*

bound of the price offer S_M can get in the described equilibrium ($p_l^{'}(s) < p_l(s) < \bar{p}(s)$). B_1 while making offers to S_M will have an absolutely continuous (conditional) distribution of offers from $p_l(s)$ to $\bar{p}(s)$, putting a mass point at $p_l(s)$.

(iii) B_1 while making offers to S_I on the equilibrium path behaves exactly in the same manner as in the two player game with one-sided asymmetric information.

(iv) Each buyer obtains a payoff of $v_B(s)$.

(Out-of-equilibrium analysis is contained in appendix (E) and (F) for public and private offers respectively.)

Proof. Suppose $\delta > \delta^*$. Then assuming existence, we first derive the equilibrium.

Define $p_l^{'}(s)$ as,

$$p_l^{'}(s) = M + \delta(1 - \alpha(s))[E_{s^{t-1}}(p) - M]$$

where $\alpha(s) = \frac{1-F(s^{t-1})}{1-F(s)}$.

This is the minimum acceptable price for S_M, when she gets only one offer. Since $E_{s^{t-1}}(p) \leq \bar{p}_{t-1}$, from lemma (19) we can say that $\bar{p}(s) > p_l^{'}(s)$.

Suppose there exists a $p_l(s) \in (p_l^{'}(s), \bar{p}(s))$ such that

$$p_l(s) = M + \delta(E_s(p) - M)$$

We can now derive the equilibrium as conjectured in the same way as we had done for the discrete types case.

Now we shall prove existence with the help of the following two lemmas.

Lemma 20 *If $s \in (s^2, H]$, then the equilibrium is identical to that of the benchmark case*

Proof. From the equilibrium of the two player game with one-sided asymmetric information we know that the buyer always offers H to the seller, who accepts it with probability 1. Thus, in the four player game, we will have an equilibrium identical to the one described in the benchmark case. ■

Lemma 21 *If there exists a $\bar{\delta} \in (\delta^{'}, 1)$ such that for $\delta \geq \bar{\delta}$ and for all $t < T (T < N^*)$ an equilibrium exists for $s \in (s^t, 1]$, then there exists a $\delta_t^* \geq \bar{\delta}$ such that, for all $\delta \in (\delta_t^*, 1)$ an equilibrium also exists for $s \in (s^{t+1}, s^t]$.*

We relegate the proof of this lemma to appendix (G).

The proof of the proposition now follows from lemma (20) and lemma (21). ■

I $E_p(\pi) > M$

Suppose not. This means that $E_p(\pi) = M$. The implication of this is that the seller S_M in equilibrium only gets offer(s) equal to M. Thus, in that case S_M can get offers from one buyer only and the offer is always equal to M. S_M always accepts this offer immediately as by rejecting she cannot get anything more. hence the equilibrium payoff of this buyer is $v - M$. The other buyer is making offers to S_I only. Since, S_M immediately accepts an offer, the payoff to her is never greater than $v_B(\pi)$ where $v_B(\pi)$ is the payoff to a buyer in the two-player game with one sided asymmetric information. As $\delta \to 1$, $v_B(\pi) \to H$. Hence, this buyer can profitably deviate by offering to S_M. Thus, we must have $E_p(\pi) > M$.

J $(v - p_2(\pi)) > v_4(\pi)$

Since $E_p(\pi) - M$ is the expected payoff to the seller S_M in equilibrium, there is at least one price offer by the buyer which is greater than or equal to $E_p(\pi)$. Hence, we must have $v_4(\pi) \leq (v - E_p(\pi))$. This gives us

$$(v - p_2(\pi)) = \delta(v - E_p(\pi)) + (1 - \delta)(v - M) > (v - E_p(\pi)) \geq v_4(\pi)$$

This concludes the proof.